Lecture 17: Bott Periodicity (Part I)

We'll follow Hatcher Chapter 2 rather closely, and these notes may be less complete than in previous lectures.

The Fundamental Product Theorem:

We'll deduce that the Bott Periodicity map

$$\beta : \tilde{K}^0(X) \longrightarrow \tilde{K}^0(S^2 X)$$

is an isom (for X cpt Hausdorff) by studying the K-theory of $X \times S^2$. We'll prove:

<u>Thm</u>: The external tensor product gives an isomorphism

$$K^0(X) \otimes K^0(S^2) \longrightarrow K^0(X \times S^2).$$
$$x \otimes y \longmapsto \pi_1^* x \otimes \pi_2^* y$$

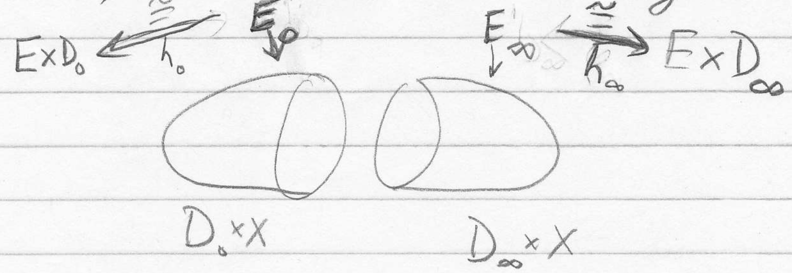
To do this we'll study bundles over $X \times S^2$ by decomposing $S^2 = \mathbb{C} \cup \{\infty\}$ into $D_0 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $D_\infty = \{z \in \mathbb{C} \mid |z| \geq 1\} \cup \{\infty\}$. The homotopy equivalences

$$X \times D_0 \simeq X \simeq X \times D_\infty$$

imply that any bundle $\begin{matrix} E' \\ \downarrow \\ X \times S^2 \end{matrix}$ restricts to a "product bundle" on both $X \times D_0$ and $X \times D_\infty$; that is if $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ is $\begin{matrix} E'|_{X \times \{1\}} \\ \downarrow \\ X \end{matrix}$, then

$$E_0 = E'|_{X \times D_0} \cong E \times D_0 \cong E'|_{X \times D_\infty} \equiv E_\infty$$

Moreover, we can reconstruct E' by clutching:



Now we have homeomorphisms

$$E' \cong \frac{E\times D_0 \amalg E\times D_\infty}{(e,z)\sim h_\infty h_0^{-1}(e,z) \text{ for } z\in D_0\cap D_\infty = S^1} \cong \frac{E\times \tilde{D}_0 \amalg E\times \tilde{D}_\infty}{(e,z)\sim \tilde{h}_\infty \tilde{h}_0^{-1}(e,z) \text{ for } z\in \tilde{D}_0\cap\tilde{D}_\infty}$$

where on the right, $\tilde{D}_0$ and $\tilde{D}_\infty$ are open neighborhoods of D_0, D_∞ (and $\tilde{h}_0, \tilde{h}_\infty$ are extensions of h_0, h_∞ to these nbhds). It's easy to check that the RH map is a homeomorphism and hence the middle space is a v. bdle. So now we can conclude all the spaces are the same.

<u>Note</u>: in this construction, we can always assume that $h_0 : E \xrightarrow{\cong} E$ ($E = E'|_{X\times\{1\}}$) and $h_\infty : E \xrightarrow{\cong} E$ are the identity: if h_0 isn't to begin with, then

$$E_0 \xrightarrow{h_0} E\times D_0 \xrightarrow[\cong]{(h_0|_E)^{-1}\times id} E\times D_0$$

gives a replacement which is.

The proof will proceed by replacing an arbitrary clutching function by simpler and simpler ones without changing the K-theory class of the resulting bundle.

<u>Key Lemma:</u> If $f: E\times S^1 \longrightarrow E\times S^1$ is a bundle isomorphism over $X\times S^1$, then
$$[E,f] := \left(E\times D_0 \amalg E\times D_\infty\right)\Big/_{\substack{(e,z)\sim f(e,z)\\ \text{for } (e,z)\in E\times S^1\subseteq E\times D_0}}$$
is a vector bundle, and if f and f' are homotopic through bundle isomorphisms, then $[E,f]\cong[E,f']$.

Moreover, if $[E,f]\cong[E,f']$, then f and f' are homotopic through bundle isomorphisms. [We will not need this second fact.]

<u>Proof:</u> $[E,f]$ is homeomorphic to
$$\left(E\times \tilde D_0 \amalg E\times \tilde D_\infty\right)\Big/_{\substack{(e,z)\sim(\pi_1 f(e,z/|z|),\, z)\\ \text{for } (e,z)\in E\times \tilde D_0\cap \tilde D_\infty\subseteq E\times D_0}}$$
where $\tilde D_0$ and $\tilde D_\infty$ are open neighborhoods of D_0 and D_∞ (not containing ∞ and 0, respectively).
(This homeomorphism is induced by the maps
$$E\times D_0 \amalg E\times D_\infty \hookrightarrow E\times \tilde D_0 \amalg E\times \tilde D_\infty$$
and
$$E\times \tilde D_0 \amalg E\times \tilde D_\infty \longrightarrow E\times D_0 \amalg E\times D_\infty$$
$$(e,z)\longmapsto \begin{cases}(e,z), & |z|\le 1\\ (e, z/|z|), & |z|>1\end{cases}$$
$$(e,z)\longmapsto \begin{cases}(e,z), & |z|\ge 1\\ (e, z/|z|) & |z|<1\end{cases}.$$
The quotient formed from $\tilde D_0$ and $\tilde D_\infty$ is a vector bundle, because it is formed by clutching along the open set $X\times(\tilde D_0\cap \tilde D_\infty)$.

Now, if f_t is a homotopy of clutching functions, then we can form the bundle homotopy

$$\xi = (E \times D_0 \times I \sqcup E \times D_\infty \times I) / (e,z,t) \sim (f_t(e,z), t)$$

and by the Bundle Homotopy Theorem, $\xi|_{X \times S^2 \times 0} = [E, f_0]$ is isomorphic to $\xi|_{X \times S^2 \times 1} = [E, f_1]$.

In the other direction say $[E,f] \cong [E,g]$.

Then $\begin{array}{c}[E,f]\\ \downarrow \\ X \times S^2\end{array}$ has trivializations h_0, h_0' over $X \times D_0$ and

h_∞, h_∞' over $X \times D_\infty$ with $h_\infty^{-1} h_0 = f$, and $(h_\infty')^{-1} h_0' = g$.

Now, $h_0^{-1} h_0' : D_0 \to Aut(E) = \{\varphi : E \to E \mid \varphi \text{ is a bundle automorphism}\}$ and $h_\infty' h_\infty^{-1} : D_\infty \to Aut(E)$ are nullhomotopic, b/c D_0, D_∞ are contractible.

Hence $g = h_\infty'(h_\infty^{-1} h_\infty)(h_0^{-1} h_0)(h_0')^{-1} = (h_\infty' h_\infty^{-1})(h_\infty h_0^{-1})(h_0 h_0'^{-1})$

$$\underset{\text{htpy through bdle auto's.}}{\simeq} h_\infty h_0^{-1} = f. \ \square$$

Example: Say $X = pt$. We want to describe the tautological bdle $\begin{array}{c}\gamma_1^1 = H\\ \downarrow \\ S^2 = \mathbb{C}P^1\end{array}$ via clutching, so we'll need to find trivializations of $\begin{array}{c}H|_{D_0}\\ \downarrow \\ D_0\end{array}$ and $\begin{array}{c}H|_{D_\infty}\\ \downarrow \\ D_\infty\end{array}$.

We have $D_0 = \{ [u,w] \in \mathbb{C}P^1 \mid |\frac{u}{w}| \leq 1 \}$

$D_\infty = \{ [u,w] \in \mathbb{C}P^1 \mid |\frac{u}{w}| \geq 1 \}$

and we have sections

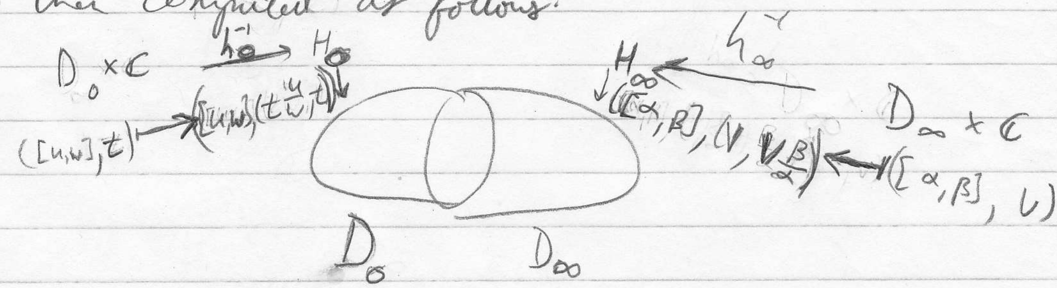
$$D_0 \longrightarrow H|_{D_0}$$
$$[u,w] \longmapsto \left([u,w], \left(\tfrac{u}{w}, 1\right)\right) \quad \text{(continuous, since } w \neq 0)$$

$$D_\infty \longrightarrow H|_{D_\infty}$$
$$[u,w] \longmapsto \left([u,w], \left(1, \tfrac{w}{u}\right)\right) \quad \text{(continuous, since } u \neq 0).$$

These sections are clearly nowhere-zero, and hence trivialize $H|_{D_0}$ and $H|_{D_\infty}$. The clutching fcn

$$f : \mathbb{C} \times S^1 \longrightarrow \mathbb{C} \times S^1$$

is then computed as follows:



So the composite $h_\infty h_0^{-1}$ sends

$$([u,w], t) \longmapsto \left([u,w], \left(t\tfrac{u}{w}, t\right)\right) \longmapsto \left([u,w], t\tfrac{u}{w}\right)$$

$|\frac{u}{w}| = 1$

Letting $z = \frac{u}{w}$, this clutching fcn is $S^1 \times \mathbb{C} \longrightarrow S^1 \times \mathbb{C}$, $(z, t) \longmapsto (z, zt)$, which we write simply as $f(z) = z$ as a map $S^1 \to GL_1\mathbb{C}$.

Arithmetic of Clutching Functions

1) If $f: E_1 \times S^1 \to E_1 \times S^1$, $g: E_2 \times S^1 \to E_2 \times S^1$ are clutching fcns,

then $f \oplus g : E_1 \oplus E_2 \times S^1 \to E_1 \oplus E_2 \times S^1$ is a clutching fcn,

and $[E_1, f] \oplus [E_2, g] \cong [E_1 \oplus E_2, f \oplus g]$ as bdles over $X \times S^2$.

2) If $f: E \times S^1 \to E \times S^1$ is a clutching fcn,

and $g: S^1 \to GL_1 \mathbb{C} = \mathbb{C}^*$ is any map, then

$$fg : E \times S^1 \to E \times S^1$$
$$(e, z) \mapsto (\pi_1 f(e,z) \cdot g(z), z)$$

is a clutching fcn, and $[E, fg] \cong [E, f] \otimes [\varepsilon^1, g]$

where $[\varepsilon^1, g]$ is the line bdle over $X \times S^2$ clutched from the

trivial line bdle over $X \times S^1$ via g.

<u>Key Computation</u>: (Ex. 1.13)

$$H \otimes H \oplus 1 \cong H \oplus H.$$

<u>Pf</u>: By 2), $H \otimes H = [\mathcal{E}^1, z] \otimes [\mathcal{E}^1, z] = [\mathcal{E}^1, z^2]$

and by 1), $[\mathcal{E}^1, z^2] \oplus \mathcal{E}^1 = [\mathcal{E}^1, z^2] \oplus [\mathcal{E}^1, 1] \cong [\mathcal{E}^1 \oplus \mathcal{E}^1, z^2 \oplus 1]$

In other words, $H \otimes H \oplus 1$ is formed by clutching a trivial 2-plane bdle via the matrix $\begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix}$.

But we have a htpy

through $P_t \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} P_t^{-1} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$ where P_t is a path from I to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} z^2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix},$$

so by the lemma

$$H \otimes H \oplus 1 \cong [\mathcal{E}^2, z^2 \oplus 1] \cong [\mathcal{E}^2, z \oplus z] \cong [\mathcal{E}^1, z] \oplus [\mathcal{E}^1, z] = H \oplus H.$$

We'll actually show that this equation completely determines $K^0(S^2)$; that is

$$K^0(S^2) \cong \mathbb{Z}[H] / (H^2 - 2H + 1) = \mathbb{Z}[H] / (H-1)^2.$$

This will actually be an important part of our proof of the product Theorem $K^0(S^2) \otimes K^0(X) \cong K^0(X \times S^2)$:

we'll show that $K^0(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K^0(X) \otimes K^0(S^2) \to K^0(X \times S^2)$ is an isom for any X.

Laurent Polynomial Clutching Functions:

We call a clutching function $\ell : E \times S^1 \to E \times S^1$ a L.P.C.F. if

$$\ell(x, z) = \left(\sum_{i=-n}^{n} a_i(x) z^i ,\ z \right)$$

for some (fiber-wise linear) endomorphisms $a_i : E \to E$.

Here $a_i(x) \cdot z^i : E_x \to E_x$ is just $a_i(x) : E_x \to E_x$ multiplied by $z \in \mathbb{C}$.

We will show that any bdle $\begin{matrix} E' \\ \downarrow \\ X \times S^2 \end{matrix}$, there is some Laurent Poly. ℓ s.t. $E' \cong [E, \ell]$.

<u>Basic Idea</u>: $E' \cong [E, f]$ for some f, and we can approximate f by some partial sum of its "Fourier Series". Then we will connect f to this Laurent poly. by a linear htpy.

We can then reduce any Laurent poly. to a linear function via stabilization (Hatcher 2.6)