Our goal is to show that Ch is an isomorphism whenever X is a finite CW cplx. We need two basic facts from K-theory:

1) **Exact Sequence** (Hatcher VB Prop. 2.9)
If $A \subset X$ is closed (X cpt Hausdorff) then the sequence $A \xhookrightarrow{i} X \xrightarrow{q} X/A$ induces an exact sequence

$$\tilde{K}^0(X/A) \xrightarrow{q^*} \tilde{K}^0(X) \xrightarrow{i^*} \tilde{K}^0(A)$$

2) **Bott Periodicity** (Hatcher VB Thm 2.11)

There are natural isomorphisms

$$\beta : \tilde{K}^0(X) \xrightarrow{\cong} \tilde{K}^0(\underbrace{S^2}_{\text{(unreduced) double suspension}} X)$$

for all compact Hausdorff space X. The map β is essentially tensor product with $[\gamma_1^1]-1$, where γ_1^1 is the tautological bdle over $S^2 \cong \mathbb{C}P^1$.
2) will allow us to compute Ch for spheres; and using 1) we can extend to all finite CW complexes.

Proof of 1 (Sketch):

- $i^*q^* = (qi)^*$, but qi is constant so $(qi)^*$ is 0 on $\tilde{K}^0$.
- To show that $\ker(i^*) \subseteq \mathrm{Im}(q^*)$, first note that $\tilde{K}^0(Z)$ is just $\mathrm{Vect}(Z)/_{\text{stable isom.}}$, so we

just want to show that if i^*E is stably trivial, then i^*E is stably isomorphic to some q^*V.

Since $i^*E = E|_A$ is stably trivial, we have $(E\oplus\varepsilon^n)|_A \cong \varepsilon^m$ for some n and m. We'll show that since this bdle is trivial over A, it is in fact the pullback of a bdle over X/A.

In fact, say E is a bdle over X which is trivial over A. Fixing a trivialization φ over A, we can identify the various fibers over A to form a quotient space $\begin{array}{c}\overline{E}\\ \downarrow\\ X/A\end{array} = E/\varphi^{-1}(a,v)\sim\varphi^{-1}(a',v)$.

Topologically, we have

$$\begin{tikzcd}
E \arrow[r, "\tilde{q}"] \arrow[d] & \overline{E} \arrow[d] \\
X \arrow[r, "q"] & X/A
\end{tikzcd}$$

and $\tilde{q}$ is a continuous fiber-wise isomorphism.

It remains only to check that $\begin{array}{c}\overline{E}\\ \downarrow\\ X/A\end{array}$ is locally trivial on a nbd of the basept $A/A \in X/A$. If A is a subcomplex of a CW cplx X, then one has a nbdh $U\supset A$ which deformation retracts to A. Then the BHT $\Rightarrow$ $E|_U$ is isomorphic to $r^*E|_A \cong \varepsilon^m$ (r the retraction $U\to A$). Now the trivialization of $E|_U$ descends to a trivialization of $\overline{E}|_{U/A}$; note that U/A is open in X/A. (Check: $U\times\mathbb{C}^n/_{(a,v)\sim(a',v)} \cong U/A\times\mathbb{C}^n$). □

We need to understand the Bott Periodicity map

$$\beta : \tilde{K}^0 X \longrightarrow \tilde{K}^0(\underset{\uparrow \text{unreduced suspension}}{S^2 X}).$$

This will be the composite

$$\tilde{K}^0 X \longrightarrow \tilde{K}^0(S^2) \otimes \tilde{K}^0 X \xrightarrow{\tilde{\mu}} \tilde{K}^0(S^2 X)$$

$$\alpha \longmapsto ([\gamma_1^1]-1) \otimes \alpha$$

$\boxed{\begin{matrix}\gamma_1^1 \\ \downarrow \\ S^2 = \mathbb{C}P^1\end{matrix} \text{ is the tautological bdle}}$

where $\tilde{\mu}$ built out of the map

$$K^0 X \otimes K^0 Y \xrightarrow{\mu} K^0(X \times Y)$$

$$[V] \otimes [W] \longmapsto [\pi_1^* V \otimes \pi_2^* W].$$

We need to replace $S^2 \times X$ with $S^2 X \simeq \underset{\uparrow \text{reduced suspension}}{S^2 \wedge X} = S^2 \times X / S^2 \vee X$

<u>Claim</u>: 1) The image under μ of $\tilde{K}^0(X) \otimes K^0(Y)$

lies in $\text{Im}(\tilde{K}^0(X \wedge Y) \longrightarrow \tilde{K}^0(X \times Y))$.

2) The map $\tilde{K}^0(X \wedge Y) \xrightarrow{\pi^*} \tilde{K}^0(X \times Y)$ is <u>injective</u>.

We can now define

$$\tilde{K}^0(X) \otimes \tilde{K}(Y) \xrightarrow{\tilde{\mu}} \tilde{K}^0(X \wedge Y)$$

to simply be the map $\mu : K^0(X) \otimes K^0(Y) \longrightarrow K^0(X \times Y)$

restricted to $\tilde{K}^0(X) \otimes \tilde{K}^0(Y)$ and considered as

a map into $\tilde{K}^0(X \wedge Y) \hookrightarrow \tilde{K}^0(X \times Y)$.

Proof of Claim: First, note that $\tilde{K}^0(X)\otimes\tilde{K}^0(Y)\hookrightarrow K^0X\otimes K^0Y\to K^0(X\times Y)$ has image in $\tilde{K}^0(X\times Y)$ (by a simple calculation).

We have the short exact sequence

$$\tilde{K}^0(X\wedge Y)\xrightarrow{q^*}\tilde{K}^0(X\times Y)\xrightarrow{i^*}\tilde{K}^0(X\vee Y)$$

so to show that $\mathrm{Im}(\tilde{K}^0(X)\otimes\tilde{K}^0(Y)\to\tilde{K}^0(X\times Y))$ lands in $\mathrm{Im}\,q^*$, we just need to check that $i^*(\alpha_1 * \alpha_2)=0\in\tilde{K}^0(X\vee Y)$. Note that the maps

$$X\leftrightarrows X\vee Y\leftrightarrows Y=(X\vee Y)/X$$

give splittings of

$$\tilde{K}^0(Y)\to\tilde{K}^0(X\vee Y)\to\tilde{K}^0(X),$$

so the restriction maps
$$i_X^*:\tilde{K}^0(X\vee Y)\to\tilde{K}^0(X)$$
$$i_Y^*:\tilde{K}^0(X\vee Y)\to\tilde{K}^0(Y)$$

give an isomorphism $\tilde{K}^0(X\vee Y)\xrightarrow[\cong]{i_X^*\oplus i_Y^*}\tilde{K}^0(X)\oplus\tilde{K}^0(Y)$.

So $i^*(\alpha_1 * \alpha_2)=0\in\tilde{K}^0(X\vee Y)\iff$ the restriction of $i^*(\alpha_1*\alpha_2)$ to X and to Y are both trivial. But we have a diagram

$$\begin{tikzcd}
\tilde{K}^0(X)\otimes\tilde{K}^0(Y) \arrow[r, "*"] \arrow[dd] & \tilde{K}^0(X\times Y) \arrow[dr] \arrow[dd] & \\
 & & \tilde{K}^0(X\vee Y) \arrow[dl] \\
\tilde{K}^0(X)\otimes\underbrace{\tilde{K}^0(y_0)}_{\text{trivial group}} \arrow[r] & \tilde{K}^0(X\times\{y_0\}) &
\end{tikzcd}$$

← basept in Y

which gives $i^*(\alpha_1 \cancel{*}\alpha_2)\big|_X = 0$, and similarly for Y.

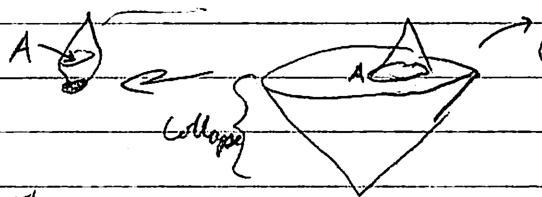
To prove 2), we need to extend the SES

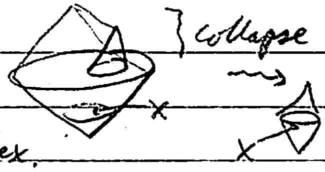
$$\tilde{K}^0(X\wedge Y) \to \tilde{K}^0(X\times Y) \to \tilde{K}^0(X\vee Y)$$

to the left. This is done via the "Puppe Sequence"

$$A \hookrightarrow X \to X/A \simeq X\cup CA \hookrightarrow \underset{\substack{\wr\Vert \\ SA}}{X\cup CA\cup CX} \to SX \simeq (X\cup CA\cup CX)\cup C(X\cup CA)$$

collapse $C(X\cup CA)$





The collapse maps are htpy equiv's if $A\subset X$ is a subcomplex. Each 3-term piece is of the form $B \hookrightarrow Z \hookrightarrow B/Z$ so we

SES associated to $X\cup CA \hookrightarrow X\cup CA\cup CX \to \frac{X\cup CA\cup CX}{X\cup CA} \cong SX$

$$\begin{tikzcd}
\tilde{K}^0\left(\frac{X\cup CA\cup CX}{X\cup CA}\right) \arrow[r] & \tilde{K}^0(X\cup CA\cup CX) \arrow[r] & \tilde{K}^0(X\cup CA) & & \\
\tilde{K}^0(SX) \arrow[u, "\wr\Vert"] \arrow[r] & \tilde{K}^0(SA) \arrow[u, "\cong"] \arrow[r] \arrow[d, equal] & \tilde{K}^0(X/A) \arrow[u, "\cong"] \arrow[r] \arrow[d, "\cong"] & \tilde{K}^0(X) \arrow[r] & \tilde{K}^0(A) \\
 & \tilde{K}^0((X\cup CA)/X) \arrow[r] & \tilde{K}^0(X\cup CA) \arrow[ur] & &
\end{tikzcd}$$

SES for $X \hookrightarrow X\cup CA \to (X\cup CA)/X \simeq SA$

Specializing, we have

$$\begin{tikzcd}
\tilde{K}^0(S(X\times Y)) \arrow[r] & \tilde{K}^0(S(X\vee Y)) \arrow[r, "⓪"] \arrow[d, dash] & \tilde{K}^0(X\wedge Y) \arrow[r] & \tilde{K}^0(X\times Y) \arrow[r] & \tilde{K}^0(X\vee Y) \\
 & \tilde{K}^0(SX\vee SY) \arrow[d, equal] & & & \\
 & \tilde{K}^0(SX)\oplus\tilde{K}^0(SY) \arrow[uul, "\text{splitting}"] & & &
\end{tikzcd}$$

The splitting $\Rightarrow \tilde{K}^0(S(X\vee Y)) \xrightarrow{0} \tilde{K}^0(X\wedge Y)$

$\Rightarrow \tilde{K}^0(X\wedge Y) \hookrightarrow \tilde{K}^0(X\times Y)$

as desired. □

Lecture 16

We're trying to show that

$$Ch: K^0 X \otimes \mathbb{Q} \to H^{even}(X; \mathbb{Q})$$

is an isomorphism whenever X is a finite CW cplx.

We'll start with the case of spheres:

Prop'n: $ch: K^0 S^{2n} \otimes \mathbb{Q} \to H^{even}(S^{2n}; \mathbb{Q})$ is an isomorphism.

Proof: We'll show that the reduced Chern character

$$(1) \qquad ch: \tilde{K}^0(S^{2n}) \to \tilde{H}^{even}(S^{2n}; \mathbb{Q}) \quad (\cong H^{2n}(S^{2n}; \mathbb{Q}) \text{ if } n>0)$$

in injective, with image $\tilde{H}^{even}(S^{2n}; \mathbb{Z}) \subseteq \tilde{H}^{even}(S^{2n}; \mathbb{Q})$. Then up to isomorphism, (1) is just the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$, and after tensoring the LHS with $\mathbb{Q}$ we get an isomorphism. Since ch is clearly an isomorphism when the space in question is a point, the diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{K}^0(S^{2n}) \otimes \mathbb{Q} \arrow[r] \arrow[d, "ch", "\cong"'] & K^0 S^{2n} \otimes \mathbb{Q} \arrow[r, two heads] \arrow[d, "ch"] & K^0(pt) \otimes \mathbb{Q} \arrow[r] \arrow[d, "ch"', "\cong"] & 0 \\
0 \arrow[r] & \tilde{H}^{even}(S^{2n}; \mathbb{Q}) \arrow[r] & H^{even}(S^{2n}; \mathbb{Q}) \arrow[r] & H^0(pt; \mathbb{Q}) \arrow[r] & 0
\end{tikzcd}$$

completes the proof. (Note that the top row remains exact after tensoring with $\mathbb{Q}$ b/c $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.)

We'll now study (1) by induction on n, using Bott periodicity. For $n=0$, (1) is clearly just $\mathbb{Z} \hookrightarrow \mathbb{Q}$.

Assuming the result for S^{2n}, we have the diagram

(2)
$$\begin{tikzcd}
\tilde{K}^0(S^{2n}) \arrow[r, "\beta", "\cong"'] \arrow[d, "ch"] & \tilde{K}^0(S^2(S^{2n})) \arrow[d, "ch"] \\
\tilde{H}^{even}(S^{2n};\mathbb{Q}) \arrow[r, "\sigma", "\cong"'] & \tilde{H}^{even}(S^2(S^{2n});\mathbb{Q})
\end{tikzcd}$$

note that $S^2(S^{2n}) = S^{2n+2}$

Where the bottom map arrises from the cross product

$$H^2(S^2;\mathbb{Q}) \otimes H^{2n}(S^{2n};\mathbb{Q}) \xrightarrow[\text{isom by Kunneth Thm}]{\cong} H^{2n+2}(S^2 \times S^{2n};\mathbb{Q})$$

in exactly the same way that β arrises from tensor product.

To be precise, the long exact sequence for $S^2 \vee S^{2n} \subseteq S^2 \times S^{2n}$

gives
$$\begin{tikzcd}
\alpha \quad \beta \arrow[rr, mapsto] & & \pi_1^*\alpha \cup \pi_2^*\beta & \\
H^2S^2 \otimes H^{2n}S^{2n} \arrow[rr] \arrow[dr, dashed, "m"] & & H^{2n+2}(S^2 \times S^{2n}) \arrow[r] & \underbrace{H^{2n+2}(S^2 \vee S^{2n})}_{0} \\
& H^{2n+2}(S^2 \wedge S^{2n}) \arrow[ur, "\cong"] & & \\
H^{2n+3}(S^2 \vee S^{2n}) = 0 \arrow[ur] & & &
\end{tikzcd}$$

and σ is just $m(c_1\gamma_1' \otimes -)$ where $c_1\gamma_1' \in H^2S^2$ is the canonical generator.

Diagram (2) commutes because

$$ch(\beta x) = ch((\gamma_1'-1) \otimes x) = ch(\gamma_1'-1) \cup ch(x)$$
$$= (ch(\gamma_1')-1) \cup ch\,x = (e^{c_1\gamma_1'}-1) \cup ch(x)$$
$$\overset{\boxed{b/c\, (c_1\gamma_1)^2 = 0}}{=} c_1\gamma_1' \cup ch(x) = \sigma(ch(x)).$$

It immediately follows that $ch : \tilde{K}^0(S^{2n+2}) \to \tilde{H}^{even}(S^{2n+2};\mathbb{Q})$ is injective, and σ restricts to an isomorphism on the subgroups

$$\tilde{H}^{even}(-;\mathbb{Z}) \subset \tilde{H}^{even}(-;\mathbb{Q});$$

which shows that $ch(\tilde{K}^0 S^{2n+2}) = \tilde{H}^{even}(S^{2n+2};\mathbb{Z}) = H^{2n+2}(S^{2n+2};\mathbb{Z})$. □

Note: There's no real need to use rational coeff's in this proof, because our computation showed that $ch(\beta(x))$ always lies in $\tilde{H}^{2n+2}(S^{2n+2};\mathbb{Z})$.

Corollary: For any sphere S^k, the Chern character gives an isomorphism $K^*(S^k) \otimes \mathbb{Q} \xrightarrow[ch]{\cong} H^*(S^k;\mathbb{Q})$.

Proof: For k even, this is the prop'n. For k odd, $(k=2n+1)$ we have $K^0(S^{2n+1}) \cong K^0(S^{2n-1}) \cong \cdots \cong K^0(S^1) \cong \mathbb{Z}$, b/c complex v. bdles over S^1 are trivial (they're classified by maps $S^1 \to Gr_n \mathbb{C}^\infty = BU(n)$, and $\pi_1 BU(n) = \pi_0 U(n) = 0$).

So $ch: \underbrace{K^0 S^{2n+1} \otimes \mathbb{Q}}_{\mathbb{Q}} \to H^{even}(S^{2n+1};\mathbb{Q}) = H^0(S^{2n+1};\mathbb{Q}) = \mathbb{Q}$

is an isomorphism (it just records the dim'n of a trivial bdle).

Next, we defined ch on K^1 by:

$$\begin{tikzcd}
K^1(S^{2n+1}) \arrow[r, dashed, "ch"] \arrow[d, equal] & \tilde{H}^{odd}(S^{2n+1};\mathbb{Q}) \arrow[d, "\cong"] \\
\tilde{K}^0(S^{2n+2}) \arrow[r, "ch"] & \tilde{H}^{even}(S^{2n+2};\mathbb{Q})
\end{tikzcd}$$

So since the bottom map is an isom after tensoring w/ $\mathbb{Q}$, so is the top one. □

We can now prove the main theorem.

Theorem: If X is a finite CW cplx, then

$$Ch: K^*(X) \otimes \mathbb{Q} \longrightarrow H^*(X;\mathbb{Q})$$

is an isomorphism.

Proof: First, note that it suffices to prove that $\tilde{K}^0(SY) \otimes \mathbb{Q} \xrightarrow[ch]{\cong} \tilde{H}^{even}(SY;\mathbb{Q})$ for suspensions, because we have the diagram

$$\begin{tikzcd} \tilde{K}^0(X) \arrow[r, "\beta", "\cong"'] \arrow[d, "ch"] & \tilde{K}^0(S^2X) \arrow[d, "ch"] \\ \tilde{H}^{even}(X;\mathbb{Q}) \arrow[r, "\cong"] & \tilde{H}^{even}(S^2X;\mathbb{Q}) \end{tikzcd}$$

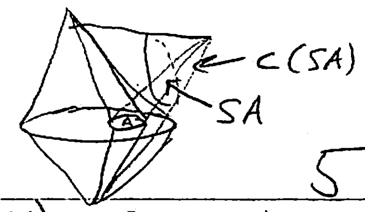
so if the RH vertical map is an isom., so is the left.

It then follows that the unreduced Chern character $K^0X \to H^{even}(X;\mathbb{Q})$ is an isom. as well. Finally, on K' we have

$$K'(X) \otimes \mathbb{Q} \cong \tilde{K}^0(SX) \otimes \mathbb{Q} \xrightarrow{ch} \tilde{H}^{even}(Sk) \cong H^{odd}(X)$$

so if ch is an isom on $\tilde{K}^0(SX)$, it's an isom. on $K'X$.

Now we prove the result by induction on the number of cells in X. When X has one cell, $SX = *$ and the result holds. If X has $n>1$ cells, then choose a subcomplex $A \subset X$ s.t. $X = A \cup e_n$, so $X/A = S^n$ and $SX/SA = S^{n+1}$. So by induction the result holds for SA and for SX/SA. Now consider the



Puppe sequence

$$\begin{tikzcd}
 & & & SX\cup C(SA) \arrow[r, two heads] \arrow[d, phantom, "\cong"] & \frac{SX\cup C(SA)}{SX} \arrow[d, equal] \\
X/A \arrow[r] \arrow[d, phantom, "\cong"] & SA \arrow[r] \arrow[d, phantom, "\cong"] & SX \arrow[r] \arrow[ur, hook] \arrow[d, equal] & SX/SA \arrow[r] & S^2A. \\
X\cup CA \arrow[r, phantom, "\subseteq"] & (X\cup CA)\cup CX \arrow[r, two heads] & \frac{(X\cup CA)\cup CX}{X\cup CA} & &
\end{tikzcd}$$

Applying $\tilde{K}^0$ and the Chern character yields

$$\begin{tikzcd}
\tilde{K}^0(S^2A) \arrow[r] \arrow[d, "ch"] & \tilde{K}^0(SX/SA) \arrow[r] \arrow[d, "ch"] & \tilde{K}^0(SX) \arrow[r] \arrow[d, "ch"] & \tilde{K}^0(SA) \arrow[r] \arrow[d, "ch"'] & \tilde{K}^0(X/A) \arrow[d, "ch"] \\
\tilde{H}^{even}(S^2A;\mathbb{Q}) \arrow[r] & \tilde{H}^{even}(SX/SA;\mathbb{Q}) \arrow[r] & \tilde{H}^{even}(SX;\mathbb{Q}) \arrow[r] & \tilde{H}^{even}(SA;\mathbb{Q}) \arrow[r] & \tilde{H}^{even}(X/A;\mathbb{Q}).
\end{tikzcd}$$

The bottom row may be constructed out of the Puppe sequence by applying $\tilde{H}^{even}$, ie. by piecing together the 3-term exact sequences of the various pairs. (This does produce the ordinary LES of the pair (SA, SX), but we don't need to worry about that.)

Tensoring the top sequence with $\mathbb{Q}$ preserves exactness and makes the four outer vertical maps isom's; note that $S^2A \simeq \Sigma^2 A$ is a CW complex with the same number of cells as A.

□