We want to reinterpret this formula in such a way that it extends to all vector bdles $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$.

Grouping terms in the previous formula gives

$$ch(E) = n + (c_1L_1 + \cdots + c_1L_n) + \frac{(c_1L_1)^2 + \cdots + (c_1L_n)^2}{2!}$$
$$+ \cdots + \frac{(c_1L_1)^k + \cdots + (c_1L_n)^k}{k!} + \cdots$$

On the other hand, the total Chern class of E is

$$c(E) = c(L_1)\cdots c(L_n) = (1 + c_1L_1)\cdots(1 + c_1L_n)$$
$$= 1 + \underbrace{\sigma_1(c_1L_1, \ldots, c_1L_n)}_{c_1E} + \cdots + \underbrace{\sigma_n(c_1L_1, \ldots, c_1L_n)}_{c_nE}$$

where σ_i denotes the i^{th} elementary symmetric polynomial:

$$\sigma_i(x_1, \ldots, x_n) = \sum_{1 \le j_1 < \cdots < j_i \le n} x_{j_1} \cdots x_{j_i}.$$

Fact: There exist polynomials $S_k(x_1, \ldots, x_k)$ $(k = 1, 2, \ldots)$ such that $\boxed{\text{For any } n}$, $S_k(\sigma_1(x_1, \ldots, x_n), \ldots, \sigma_k(x_1, \ldots, x_n)) = x_1^k + \cdots + x_n^k$

Although a priori, there could be a dependence on n.

These are called the Newton Polynomials, and their existence is guaranteed by the basic theory of symmetric polynomials: any symm. poly of degree k (e.g. $x_1^k + \cdots + x_n^k$) is a poly in $\sigma_1(x_1, \ldots, x_n), \ldots, \sigma_k(x_1, \ldots, x_n)$. We'll construct S_k explicitly.

We now have $S_k(\overbrace{\sigma_1(c_1L_1, \ldots, c_1L_n)}^{c_1E}, \ldots, \overbrace{\sigma_n(c_1L_1, \ldots, c_1L_n)}^{c_nE}) = (c_1L_1)^k + \cdots + (c_1L_n)^k$

So $ch(E) = \underset{\boxed{n = \dim E}}{n} + \sum c_1L_i + \frac{\sum (c_1L_i)^2}{2!} + \cdots = n + S_1(c_1E) + S_2(c_1E, c_2E)/2! + \cdots$

<u>Def'n</u>: For any vector bdle $\begin{smallmatrix}E\\ \downarrow\\ X\end{smallmatrix}$ w/ X fin. dim'l,

$$ch(E) = \dim(E) + \sum_{k>0} s_k(c_1E, \dots, c_kE)/k! \in H^*(X;\mathbb{Q})$$

What are the Newton Polynomials?

- $s_1(x) = x$; so $s_1(\sigma_1(x_1, \dots, x_n)) = s_1(x_1 + \dots + x_n) = x_1 + \dots + x_n$, as desired.
- $s_2(x_1, x_2) = x_1^2 - 2x_2$;

so $s_2(\sigma_1(x_1, \dots, x_n), \sigma_2(x_1, \dots, x_n)) = (x_1 + \dots + x_n)^2 - 2\sum_{i<j} x_i x_j$

$$= x_1^2 + \dots + x_n^2 + \sum_{i\neq j} x_i x_j - 2\sum_{i<j} x_i x_j$$

$$= x_1^2 + \dots + x_n^2,$$

as desired.

To construct s_k in general, we start with the following observation:

$$(t - x_1) \cdots (t - x_k) = \sum_{i=0}^{k} (-1)^{k-i} t^i \sigma_{k-i}(x_1, \dots, x_k),$$

and setting $t = x_j$ yields

$$0 = \sum_{i=0}^{k} (-1)^{k-i} x_j^i \sigma_{k-i}(x_1, \dots, x_k)$$

i.e.

$$x_j^k = \sum_{i=0}^{k-1} (-1)^{k-i+1} x_j^i \sigma_{k-i}(x_1, \dots, x_k)$$

and hence (summing over $j = 1, \dots, k$)

$$(\ast) \qquad x_1^k + \dots + x_k^k = \sum_{i=0}^{k-1} (-1)^{k-i+1} (x_1^i + \dots + x_k^i)\, \sigma_{k-i}(x_1, \dots, x_k)$$

Since the RHS can be expressed in terms of the earlier Newton polynomials $s_0 = 1, s_1, \dots, s_{k-1}$, we obtain a recursive def'n for s_k, which we write in terms of new variables t_i (which we will substitute with $\sigma_i(x_1, \dots, x_n)$ shortly).

<u>Def'n</u> $s_k(t_1, \dots, t_k) = \left(\sum_{i=1}^{k-1} (-1)^{k-i+1} s_i(t_1, \dots, t_i)\, t_{k-i}\right) + (-1)^{k+1} k t_k$;

the recursion begins with $s_1(t_1) = 1$.

Lecture 15 We now need to prove the Fact:

<u>Prop'n:</u> The polynomials $s_k(t_1, \dots, t_k)$ defined above satisfy:

$$s_k(\sigma_1(x_1, \dots, x_n), \dots, \sigma_k(x_1, \dots, x_n)) = x_1^k + \dots + x_n^k$$

for <u>any</u> $n \geq 1$.

<u>Pf:</u> By induction.

For $k = 1$, both sides are $\sum_{i=1}^{n} x_i$. Assuming the result for $l < k$, we have (when $\underline{\underline{n = k}}$)

$$s_k(\sigma_1(x_1, \dots, x_k), \dots, \sigma_k(x_1, \dots, x_k)) = \sum_{i=1}^{k-1} (-1)^{k-i+1} s_i(\sigma_1(x_1, \dots, x_k), \dots, \sigma_i(x_1, \dots, x_k))\, \sigma_{k-i}(x_1, \dots, x_k)$$
$$+ (-1)^{k-1} k\, \sigma_k(x_1, \dots, x_n)$$

$$\underset{\text{by induction}}{=} \left(\sum_{i=1}^{k-1} (-1)^{k-i+1} (x_1^i + \dots + x_k^i)\, \sigma_{k-i}(x_1, \dots, x_k)\right) + (-1)^{k+1} k\, \sigma_k(x_1, \dots, x_k)$$

$$= \sum_{i=0}^{k-1} (-1)^{k-i+1} (x_1^i + \dots + x_k^i)\, \sigma_{k-i}(x_1, \dots, x_k)$$

$$\underset{\text{by eq'n } (\star)}{=} x_1^k + \dots + x_k^k.$$

So we get the desired formula, at least when $k = n$.

To prove the formula for $k>n$, we simply write

$$S_k(\sigma_1(x_1,\dots,x_n),\dots,\sigma_k(x_1,\dots,x_n)) = S_k(\sigma_1(x_1,\dots,x_n,\overbrace{0,\dots,0}^{k-n}),\dots,\sigma_k(x_1,\dots,x_n,\overbrace{0,\dots,0}^{k-n}))$$

$$\underset{\boxed{\text{by } k=n \text{ case}}}{=} x_1^k+\dots+x_n^k+0^k+\dots+0^k = x_1^k+\dots+x_n^k$$

as desired.

On the other hand, say $k<n$. Then by the theory of symmetric poly's, we know that there exists a polynomial $S_{k,n}(t_1,\dots,t_k)$ such that

$$(\star)\quad S_{k,n}(\sigma_1(x_1,\dots,x_n),\dots,\sigma_k(x_1,\dots,x_n)) = x_1^k+\dots+x_n^k.$$

Now

$$S_{k,n}(\sigma_1(x_1,\dots,x_k),\dots,\sigma_k(x_1,\dots,x_k))$$

$$= S_{k,n}(\sigma_1(x_1,\dots,x_k,\overbrace{0,\dots,0}^{n-k}),\dots,\sigma_k(x_1,\dots,x_k,\overbrace{0,\dots,0}^{n-k}))$$

$$= x_1^k+\dots+x_k^k+0^k+\dots+0^k = S_k(\sigma_1(x_1,\dots,x_k),\dots,\sigma_k(x_1,\dots,x_k)).$$

But the theory of symmetric polynomials also says that there is a unique polynomial in k variables such that

$$p(\sigma_1(x_1,\dots,x_k),\dots,\sigma_k(x_1,\dots,x_k)) = x_1^k+\dots+x_k^k.$$

Since both $S_{k,n}$ and S_k satisfy this equation, we must have $S_k = S_{k,n}$, meaning that S_k satisfies $(\star)$. $\square$

We can now show that the Chern Character is a ring homomorphism:

<u>Theorem</u>: The Function

$$\underline{\mathbf{Vect}}(X) \xrightarrow{Ch} H^*(X;\mathbb{Q})$$
$$[V] \longmapsto \dim V + \sum_{k>0} S_k(c_1V, \dots, c_kV)/k! \in \bigoplus_{i=0}^{\infty} H^{2i}(X;\mathbb{Q})$$

extends uniquely to a <u>ring</u> <u>homomorphism</u>

$$Ch: K^0(X) \longrightarrow H^*(X;\mathbb{Q}).$$

<u>Pf</u>: We are forced to define

$$Ch([V]-[W]) = \dim V - \dim W + \sum_{k>0} \frac{S_k(c_1V, \dots, c_kV) - S_k(c_1W, \dots, c_kW)}{k!},$$

and we must check that this is a ring homomorphism.

<u>Additivity</u>: We want to show that $Ch(V \oplus W) = Ch(V) + Ch(W)$ (the formula on formal differences then follows formally).

We'll use the splitting principle; that is, we want a map $f: Y \to X$ s.t. $f^*: H^*(X;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$ is injective, and f^*V, f^*W are sums of line bdles. For integer coeff's, this is achieved by iterated use of projective bdles. The proof of the proj. Bdle Thm also works with $\mathbb{Q}$-coeff's, so $H^*(E;\mathbb{Q}) \to H^*(PE;\mathbb{Q})$ is always injective, Now, it suffices to show $ch(L_1\oplus\cdots\oplus L_n) = \sum ch(L_i)$,

for any line bdles $L_1,\dots,L_n$. But this was exactly how we defined ch:

$$ch(L_1\oplus\cdots\oplus L_n) = n + \sum_{k>0} \frac{s_k(c_1(L_1\oplus\cdots\oplus L_n),\dots,c_k(L_1\oplus\cdots\oplus L_n))}{k!}$$

$$= n + \sum_{k>0} \frac{s_k(\sigma_1(c_1L_1,\dots,c_1L_n),\dots,\sigma_k(c_1L_1,\dots,c_1L_n))}{k!}$$

$$= n + \sum_{k>0} \frac{(c_1L_1)^k + \cdots + (c_1L_n)^k}{k!} = \sum_{i=1}^{n}\sum_{k=0}^{\infty} \frac{(c_1L_i)^k}{k!}$$

$$= \sum_{i=1}^{n} e^{c_1L_i}.$$

We originally defined $Ch(L_i) = e^{c_1L_i}$ for line bdles but we do need to check that this is consistent with our general def'n, i.e. that

$$1+\sum_{k=1}^{\infty} \frac{(c_1L)^k}{k!} = 1+\sum_{k=1}^{\infty} \frac{s_k(c_1L,\overbrace{c_2L,\dots,c_kL}^{\boxed{\text{all zero}}})}{k!}$$

<u>Claim</u>: For each k, $s_k(t,0,\dots,0) = t^k$.

<u>Pf</u>: By induction on k, using the recursion relation.

For $k=1$, $s_1(t) = t = t^1$. Assuming the Claim for $l<k$, we have

$$s_k(t,0,\dots,0) = (-1)^{k+1}k\cdot 0 + \sum_{i=1}^{k-2}(-1)^{k-i+1} s_i(t,0,\dots,0)\cdot \overset{t_{k-i}=0 \text{ for } i<k-1}{0} + \cancel{(-1)^{k-(k-1)+1}} s_{k-1}(t,0,\dots,0)t$$

$$\overset{\boxed{\text{induction}}}{=} (t^{k-1})\cdot t = t^k. \quad \square$$

<u>Multiplicativity</u>: We need to show that

$$ch(V \otimes W) = ch(V)\, ch(W).$$

We again can check the case of sums of line bdles, and then apply the splitting principle.

$$ch((L_1 \oplus \cdots \oplus L_n) \otimes (L_1' \oplus \cdots \oplus L_m')) = ch\Big(\bigoplus_{i,j} L_i \otimes L_j'\Big)$$

$$\overset{\boxed{\text{Additivity}}}{=} \sum_{i,j} ch(L_i \otimes L_j') = \sum_{i,j} e^{c_1(L_i \otimes L_j')}$$

$$= \sum_{i,j} e^{c_1 L_i + c_1 L_j'} = \sum_{i,j} e^{c_1 L_i} e^{c_1 L_j'}.$$

On the other hand,

$$ch(L_1 \oplus \cdots \oplus L_n)\, ch(L_1' \oplus \cdots \oplus L_m') = \Big(\sum_i ch L_i\Big)\Big(\sum_j ch(L_j')\Big)$$

$$= \sum_{i,j} (ch L_i)(ch L_j') = \sum_{i,j} e^{c_1 L_i} e^{c_1 L_j'}. \quad \square$$

We can now extend ch to a map (an additive hom, at least)

$$K^1 X = \tilde{K}^0(SX) \longrightarrow \bigoplus_{i=0}^{\infty} H^{2i+1}(X;\mathbb{Q}) \subset H^*(X;\mathbb{Q}).$$

This is achieved by the diagram

$$\begin{array}{ccc} K^0(SX) & \xrightarrow{ch} & \bigoplus H^{2i}(SX;\mathbb{Q}) \\ \downarrow & & \downarrow \\ K^0(pt) & \xrightarrow{ch} & \bigoplus H^{2i}(pt;\mathbb{Q}), \end{array}$$

which (upon taking kernels) yields the Chern Character

$$K^1 X := \tilde{K}^0(SX) \longrightarrow \bigoplus_{i \geq 0} \tilde{H}^{2i}(SX;\mathbb{Q}) \cong \bigoplus_{i \geq 0} H^{2i-1}(X;\mathbb{Q}).$$