Grothendieck's Def'n of Chern / Stiefel-Whitney Classes

$c_1(L_E)^n \in H^*PE \underset{\text{Projective Bdle Thm}}{\cong} H^*B \otimes_{\mathbb{Z}} H^*\mathbb{C}P^{n-1}$. Since $H^*\mathbb{C}P^{n-1} \cong \mathbb{Z}[\alpha]/(\alpha^n)$, there are unique $x_i \in H^{2i}(B)$ s.t.

$$c_1(L_E)^n = \sum (-1)^{n+i-1} q^*(x_{n-i}) \cup c_1(L_E)^i,$$

where $q: PE \to B$. We define $c_i(E) = x_i$ for $i = 0, \cdots, \dim(E) = n$.

We now need to check that these classes satisfy the axioms from the Thm on p. 1.

Axiom 1: Grothendieck's formula only defines the Chern/ Stiefel-Whitney classes in the dimensions where they are allowed to be non-zero. We simply extend these def'ns by setting $c_k(E) = 0$ if $k > 2\dim E$, $w_k(E) = 0$ if $k > \dim E$.

Axiom 3: If $\begin{matrix}\gamma\\ \downarrow\\ B\end{matrix}$ is a line bdle, then the projection $\begin{matrix}P(\gamma)\\ \downarrow \pi\\ B\end{matrix}$ is a homeomorphism, and we have a canonical isomorphism $\pi^*\gamma \cong L_\gamma$ of line bdles over $P(\gamma) \cong B$. Grothendieck's formula then defines

$$c_1(L_\gamma) =: c_1(L) \cdot 1_{H^*(P(\gamma);\mathbb{Z})},$$

so our two definitions of c_1 for line bdles agree (and similarly for w_1).

The Whitney Sum Formula (Axiom 2) will take some work, so first we explain naturality. We want the c_i, w_i to be characteristic classes, so we must show that in any diagram
$$\begin{tikzcd} f^*E \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$
we have $c_i(f^*(E)) = f^*(c_i E) \in H^*(B;\mathbb{Z})$ (or $w_i(f^*E) = f^*(w_i(E)) \in H^*(B;\mathbb{Z}/2)$).

This naturality property follows from the fact that projective bdles (and their tautological line bdles) behave well under pullbacks: We have a diagram

$$\begin{tikzcd}
L_{f^*E} \arrow[r, "\cong"] \arrow[d] & (Pf)^*L_E \arrow[r] \arrow[d] & L_E \arrow[d] \\
P(f^*E) \arrow[r, "\cong"] \arrow[dr] & \tilde{f}^*(PE) \arrow[r, "Pf"] \arrow[d, "q'"] & PE \arrow[d, "q"] \\
 & B' \arrow[r, "f"] & B
\end{tikzcd}$$

and now the eq'n $c_1(L_E)^n = \sum_{i=1}^{n} (-1)^{n-i+1} q^* c_i(E) \cup c_1(L_E)^{n-i}$ pulls back to give

$$\left((Pf)^*(c_1(L_E))\right)^n = \sum_{i=1}^{n} (-1)^{n-i+1} (Pf)^* q^* c_i E \cup (Pf)^*(c_1 L_E)^{n-i}.$$

Since c_1 is already natural, we have $(Pf)^*(c_1 L_E) = c_1(L_{f^*E})$, so the classes $Pf^*(c_i E) \in H^*(P f^* E) \cong \tilde{f}^*(P_E)$ must be the Chern classes of f^*E (i.e. the unique classes satisfying Grothendieck's formula).

Before proving the Whitney Sum Formula, we need to introduce two more constructions on vector bdles: tensor products and duals. We will follow MS §3, which gives a very general method for extending "continuous" functors on vector spaces to vector bdles.

<u>Def'n</u>: Let <u>Vect</u> denote the category of finite dim'l v. spaces (over $\mathbb{R}$ or $\mathbb{C}$) and <u>isomorphisms</u>. A functor $F: \underline{\text{Vect}}^k \to \underline{\text{Vect}}$ is <u>continuous</u> if each component

To prove the Whitney Sum Formula, we'll use the following lemmas:

Lemma 1: For any line bdl $\begin{matrix} L \\ \downarrow p \\ X \end{matrix}$, the line bdle $\begin{matrix} L\otimes L^* \\ \downarrow \\ X \end{matrix}$ is trivial. Here L^* is the dual bundle, constructed in the real case from the functor $V \mapsto \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) \overset{\text{def}}{=} V^*$, and in the cplx case from $V \mapsto \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$.

Pf: If $v \in p^{-1}(x)$ is any non-zero vector, then set $s(x) = v \otimes v^*$, where $v^* \in \mathrm{Hom}(p^{-1}(x),\mathbb{R})$ sends w to the unique $c \in \mathbb{R}$ s.t. $cw = v$ (i.e. c is the coordinate of w in the basis $\{v\}$). Note that if $c \neq 0$, then $(cv)\otimes(cv)^* = cv \otimes \frac{1}{c}(v^*) = v \otimes v^*$ so s is well-defined and continuous. Since s is never zero, it follows that $L \otimes L^*$ is trivial. The cplx case is the same. □

[Alternatively, $L\otimes L^* \cong \mathrm{Hom}(L,L)$, and Id_L is a section of $\mathrm{Hom}(L,L)$.]

Lemma 2: The first Chern/SW class is additive on line bdles:

$$c_1(L_1 \otimes L_2) = c_1 L_1 + c_1 L_2 \in H^1(X;\mathbb{Z})$$
$$w_1(L_1 \otimes L_2) = w_1 L_1 + w_1 L_2 \in H^1(X;\mathbb{Z}/2).$$

We postpone the proof.

Proof of the Whitney Sum Formula:

Case 1: Say $E = L_1 \oplus \cdots \oplus L_k$, a sum of line bdles.

We will show that $c(E) = (1+c_1L_1)(1+c_1L_2)\cdots(1+c_1L_k)$, as expected from iterated application of the WSF. It then follows that $c((L_1\oplus\cdots\oplus L_k)\oplus(L_1'\oplus\cdots\oplus L_\ell')) = \prod(1+c_1L_i)\cdot\prod(1+c_1L_i') = c(L_1\oplus\cdots\oplus L_k)\,c(L_1'\oplus\cdots\oplus L_\ell')$, proving the WSF for sums of line bdles.

Consider the bdle q^*E, where $\begin{matrix} P(E) \\ \downarrow q \\ X \end{matrix}$ is the projective bdle associated to E. Then there is an injective bdle map $L_E \longrightarrow q^*E$ (here $L \in P(E)$ and $l \in E$ is a point on L).

$$(L, l) \longmapsto (L, l)$$

Tensoring with L_E^* gives an injection

$$L_E \otimes L_E^* \hookrightarrow (q^*E) \otimes L_E^* \cong (q^*L_1 \oplus \cdots \oplus q^*L_k) \otimes L_E^*$$
$$\cong (q^*L_1 \otimes L_E^*) \oplus \cdots \oplus (q^*L_k \otimes L_E^*).$$

The section of $L_E \otimes L_E^*$ (Lemma 1) gives a section s of $\bigoplus_{i=1}^{k} q^*L_i \otimes L_E^*$, and projecting to the factors yields sections s_i of $q^*L_i \otimes L_E^*$. Let $V_i \subseteq P(E)$ be the open set on which s_i is non-zero. Then since $s = \bigoplus_{i=1}^{k} s_i$ is never zero, we must have $\bigcup_{i=1}^{k} V_i = P(E)$. Now, note that $(q^*L_i \otimes L_E^*)\big|_{V_i}$ is trivial, so $c_1(q^*L_i \otimes L_E^*|_{V_i}) = 0$.

By naturality of c_1, we have $c_1(q^*L_i \otimes L_E^*)\big|_{V_i} = 0$, where $|_{V_i}$ indicates the map on cohomology $H^2PE \to H^2V_i$.

By exactness of the relative cohomology sequences $H^2(PE, V_i) \xrightarrow{j_i} H^2(PE) \to H^2(V_i)$, there exist classes $\gamma_i \in H^2(PE, V_i)$ s.t. $j_i(\gamma_i) = c_1(q^*L_i \otimes L_E^*)$. The relative cup product $\gamma_1 \cup \cdots \cup \gamma_k$ lies in $H^2(PE, \bigcup_{i=1}^{k} V_i) = H^2(PE, PE) = 0$.

But for any pair $U, V \subseteq Y$, with U, V open, the diagram

$$\begin{tikzcd} H^*(Y,A)\times H^*(Y,B) \arrow[r, "\cup"] \arrow[d, "j_A\times j_B"] & H^*(Y, A\cup B) \arrow[d, "j_{A\cup B}"] \\ H^*(Y)\times H^*(Y) \arrow[r, "\cup"] & H^*(Y) \end{tikzcd} \quad (\star)$$

commutes, where the vertical maps come from the LES of the pairs.

In our case, this says that $j(\gamma_1 \cup \cdots \cup \gamma_k) = \prod (j_i \gamma_i)$
$= \prod c_1(q^* L_i \otimes L_E^*)$,

where $j: H^*(PE, PE) \longrightarrow H^* PE$. Since $H^*(PE, PE) = 0$, we have $\boxed{\prod c_1(q^* L_i \otimes L_E^*) = 0.}$ $(\star\star)$

[Aside: Commutativity of $(\star)$ follows by tracing the def'ns in Hatcher (§3.2, p. 209). The relative cup product is defined via top line in the following diagram:

$$\begin{tikzcd} C^k(Y,A)\times C^l(Y,B) \arrow[r, "\cup"] \arrow[d, "j_A\times j_B"] & C^{k+l}(Y, A+B) \arrow[d, bend right, "j_{A+B}"'] & C^{k+l}(Y, A\cup B) \arrow[l, "\text{isom on } H^*"'] \arrow[dl, "j_{A\cup B}"] \\ C^k Y \times C^l Y \arrow[r, "\cup"] & C^{k+l}(Y) & \end{tikzcd}$$

$C^{k+l}(Y, A+B) = \{\varphi : C_{k+l}(Y) \to \mathbb{Z} \mid \varphi \text{ vanishes on } C_k(A) \text{ and } C_k B\}$

where $\cup$ in both cases is given by the usual formula:

$\varphi \cup \psi\, (\sigma) = \varphi(\sigma|_{[v_0, \ldots, v_k]})\, \psi(\sigma|_{[v_{k+1}, \ldots, v_{k+l}]})$.

It's quick to check this diagram commutes, so $(\star)$ commutes too.]

Continuing, we have $c_1 L_E + c_1 L_E^* \overset{\text{Lemma 2}}{=\!=} c_1(L_E \otimes L_E^*) \overset{\text{Lemma 1}}{=\!=} 0$, So $c_1(L^*) = -c_1(L)$. Thus Equation $(\cancel{\ast}\ast\ast)$ becomes

$$\prod_{i=1}^{k} \big(\underbrace{c_1(q^*L_i)}_{\substack{\| \\ q^*(c_1L_i)}} - c_1L_E\big) = 0,$$

i.e.

$$(c_1L_E)^k = (-1)^{k+1}\Big(\sum_{\ell=1}^{k} (-1)^{k-\ell}\Big(\sum_{1\le i_1<\cdots<i_\ell\le k} q^*(c_1L_{i_1})\cup\cdots\cup q^*(c_1L_{i_\ell})\Big)(c_1L_E)^{k-\ell}\Big).$$

Hence by our def'n of Chern/Stiefel-Whitney classes, we find that

$$(-1)^{1-\ell}\sum_{1\le i_1<\cdots<i_\ell\le k} q^*(c_1L_{i_1}\cup\cdots\cup c_1L_{i_\ell}) = (-1)^{\ell+1}c_\ell(E)$$

i.e. $$c_\ell(E) = \sum_{1\le i_1<\cdots<i_\ell\le k} c_1L_{i_1}\cup\cdots\cup c_1L_{i_\ell} \in H^*(X).$$

So $c(E) = 1 + c_1E + \cdots + c_kE = \prod_{i=1}^{k}(1+c_1L_i)$ as claimed.

We now deduce the general case. Given any $\begin{smallmatrix}E\\ \downarrow\\ X\end{smallmatrix}$, the linear injection $L_E \hookrightarrow q^*E$ induces a splitting $q^*E = L_E \oplus L_E^\perp$ (see HW2) of bdles over $P(E)$. We can apply the same principle again, and we find that the pullback of $q^*L_E^\perp$ over $P(q^*E)$ splits as $L_1 \oplus L_2 \oplus E''$ with L_1, L_2 line bdles. Iterating, we find that there is a map $\tilde{X} \xrightarrow{\pi} X$ such that π^*E is a sum of line bdles, and $\pi^*: H^*X \to H^*\tilde{X}$ is a composite of injections (of the form $q^*: H^*P(\xi) \to H^*Y$ for various bdles $\begin{smallmatrix}\xi\\ \downarrow\\ Y\end{smallmatrix}$).

Now if $\begin{smallmatrix} E & & E' \\ & \searrow \ \swarrow & \\ & X & \end{smallmatrix}$ are two bdles, let $X_1 \xrightarrow{\pi_1} X$ be such a map for E, and let $X_2 \xrightarrow{\pi_2} X_1$ be such a map for $\pi_1^* E'$.

Then over X_2, we have $(\pi_2 \circ \pi_1)^* E = L_1 \oplus \cdots \oplus L_n$ and $(\pi_2 \circ \pi_1)^* E' = L_1' \oplus \cdots \oplus L_m'$ for some line bdles L_i, L_j'.

Hence by the previous case,

$$c((\pi_2 \circ \pi_1)^* E \oplus (\pi_2 \circ \pi_1)^* E') = c((\pi_2 \circ \pi_1)^* E) \cup c((\pi_2 \circ \pi_1)^* E')$$

i.e. $$(\pi_2 \circ \pi_1)^* (c(E \oplus E')) = (\pi_2 \circ \pi_1)^* (c(E) \cup c(E'))$$

But $\pi_2 \circ \pi_1^* : H^*(X) \longrightarrow H^*(X_2)$ is injective, so we have

$$c(E \oplus E') = c(E) \cup c(E') \text{ in } H^*(X). \quad \square$$

<u>Rmk</u>: The previous method is known as the Splitting Principle: heuristically, it says that if one wants to derive a formula for all bdls, one just finds a formula that works for sums of line bdles, and then checks that it extends (by the above method). The main pt. is that for every bdle $\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}$, there is a map $X' \xrightarrow{f} X$ s.t. $f^* : H^* X' \hookrightarrow H^* X$, and $f^* E$ is a sum of line bdles.

Lecture 9

Another nice application of the Splitting Principle is the uniqueness of Chern / Stiefel-Whitney classes.

<u>Theorem</u>: The classes w_i, c_i we have defined are the <u>only</u> sequences of real/cplx char. classes satisfying the 3 axioms.

<u>Pf</u>: Let $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ be a (cplx, say) bdle, and let $X' \xrightarrow{f} X$ be a map s.t. f^* is injective on cohomology and $f^*E = L_1 \oplus \cdots \oplus L_k$ for line bdles $L_1, \ldots, L_k$. Then if $\beta = 1 + \beta_1 + \cdots + \beta_k$ are char. classes satisfying the axioms, we have

$$f^*(\beta(E)) = \beta(f^*E) = \beta(L_1 \oplus \cdots \oplus L_k) \overset{WSF}{=} \prod_{i=1}^{k} \beta(L_i)$$

$$\underset{\substack{\uparrow \\ \text{axioms} \Rightarrow \\ \beta_2, \beta_3, \ldots \text{ vanish on line bdles}}}{=} \prod_{i=1}^{k} (1 + \beta_1(L_i)) \underset{\substack{\uparrow \\ \text{the axioms determine} \\ \text{values on line bdles}}}{=} \prod_{i=1}^{k} (1 + c_1(L_i)) \overset{WSF}{=} c(\oplus L_i)$$

$$= c(f^*E) = f^* c(E).$$

Since f^* is injective, we have $\beta(E) = c(E)$. $\square$

<u>Pf of Lemma 2</u>: (Real Case) If $L_1 \searrow_X \swarrow L_2$ are $\mathbb{R}$-bdles, we must show that $w_1(L_1 \otimes L_2) = w_1 L_1 + w_1 L_2$. Since we <u>defined</u> w_1 in terms of pullbacks along loops, this means we just need to check that if $N_1 \searrow_{S^1} \swarrow N_2$ are line bundles, then $N_1 \otimes N_2$ is non-trivial $\Longleftrightarrow$ exactly one of the N_i is non-trivial. The only bdles over S^1 are $\begin{matrix} \mathbb{R} \times S^1 \\ \downarrow \\ S^1 \end{matrix}$ and $\begin{matrix} \gamma_1^1 \\ \downarrow \\ S^1 = \mathbb{RP}^1 \end{matrix}$, so we just need to check that $\gamma_1^1 \otimes \gamma_1^1$ is trivial. Choosing a metric gives $\gamma_1^1 \otimes \gamma_1^1 \cong \gamma_1^1 \otimes (\gamma_1^1)^* \underset{\uparrow}{\cong} S^1 \times \mathbb{R}$. $\square$

$\gamma_1^1 \otimes (\gamma_1^1)^*$ has a section $x \mapsto u_x \otimes (u_x \mapsto 1)$ where $u_x \in (\gamma_1^1)_x$ is a unit vector.

[The last step can also be done using clutching Fcns.]

To establish additivity of the first <u>Chern</u> class, we take a more homotopy-theoretical approach. We will show that tensor product gives $\mathbb{C}P^\infty$ a multiplicative structure (up to homotopy), which then induces both tensor product of line bdles and addition of first cohomology classes. (This works equally well for w_1 and $\mathbb{R}P^\infty$.)

To construct a "multiplication" $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu} \mathbb{C}P^\infty$, it suffices to name a line bdle over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ (and then μ will be the classifying map. We will use the bdle $\pi_1^*\gamma_1 \otimes \pi_2^*\gamma_1$, where $\begin{matrix}\gamma_1\\ \downarrow \\ \mathbb{C}P^\infty\end{matrix}$ is the tautological line bdle, and
$$\begin{tikzcd} & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{C}P^\infty & & \mathbb{C}P^\infty \end{tikzcd}$$
are the projections.

<u>Claim</u>: If $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$ classifies $\pi_1^*\gamma_1 \otimes \pi_2^*\gamma_1$ (i.e. $\mu^*\gamma_1 \cong \pi_1^*\gamma_1 \otimes \pi_2^*\gamma_1$) then

1) If $\begin{matrix} i_1 : \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty \\ x \longmapsto (x, *) \end{matrix}$ and $\begin{matrix} i_2 : \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty \\ x \longmapsto (*, x) \end{matrix}$ are the inclusions, then $\mu i_1, \mu i_2 : \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty$ are homotopic to the identity (for any $* \in \mathbb{C}P^\infty$);

2) If $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ is classified by $f : X \to \mathbb{C}P^\infty$ and $\begin{matrix} M \\ \downarrow \\ X \end{matrix}$ is classified by $g : X \to \mathbb{C}P^\infty$, then $\begin{matrix} L \otimes M \\ \downarrow \\ X \end{matrix}$ is classified by $\mu \circ (f, g)$.

<u>Proof</u>: <u>1)</u> To check that $\mu i_1, \mu i_2$ are homotopic to $Id_{\mathbb{C}P^\infty}$, it suffices to check that $(\mu i_1)^* \gamma_1, (\mu i_2)^* \gamma_1$ are isomorphic to $\gamma_1 = Id^*(\gamma_1)$. We have

$$(\mu i_1)^* \gamma_1 \cong i_1^* \mu^* \gamma_1 \cong i_1^* (\pi_1^* \gamma_1 \otimes \pi_2^* \gamma_1)$$
$$\cong (\pi_1 i_1)^* \gamma_1 \otimes \underbrace{(\pi_2 i_1)}_{\text{constant}}{}^* \gamma_1 \cong (Id)^* \gamma_1 \otimes (X \times \mathbb{C})$$
$$\cong \gamma_1,$$

and similarly $(\mu i_2)^* \gamma_1 \cong \gamma_1$.

<u>Picture</u>:

$$\begin{tikzcd} & & \mathbb{C}P^\infty & \\ \mathbb{C}P^\infty \arrow[urr, "\text{constant}"] \arrow[r, hook, "i_1"] \arrow[dr, equal] & \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[ur, "\pi_2"] \arrow[d, "\pi_1"] \arrow[rr, "\mu"] & & \mathbb{C}P^\infty \\ & \mathbb{C}P^\infty & & \end{tikzcd}$$

<u>2)</u> If $f^* \gamma_1 = L$, $g^* \gamma_1 = M$, then $(\mu \circ (f,g))^* (\gamma_1) = (f,g)^* (\pi_1^* \gamma_1 \otimes \pi_2^* \gamma_1)$

$$= (\pi_1 \circ (f,g))^* \gamma_1 \otimes (\pi_2 \circ (f,g))^* \gamma_1 = f^* \gamma_1 \otimes g^* \gamma_1 = L \otimes M. \quad \square$$

To compute $c_1(L \otimes M)$, we need to know a bit about the cohomology of $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$. The following result is a weak form of the Kunneth Thm, and is proven in Hatcher (Chap. 3, p. 218-223).

<u>Kunneth Thm</u>: If X, Y are CW cplxs with $H^*(Y;\mathbb{Z})$ torsion-free, then the map
$$H^*(X;\mathbb{Z}) \times H^*(Y;\mathbb{Z}) \longrightarrow H^*(X \times Y; \mathbb{Z})$$
$$\alpha \times \beta \longmapsto \pi_1^* \alpha \cup \pi_2^* \beta$$
induces an isomorphism
$$H^*(X;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z}) \xrightarrow{\cong} H^*(X \times Y; \mathbb{Z}).$$

Since $\mathbb{C}P^\infty$ is a CW cplx and $H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[\alpha]$, the theorem applies in our setting.

<u>Theorem</u>: If $L \searrow X \swarrow M$ are line bdles over a paracpt space, then $c_1(L\otimes M) = c_1 L + c_1 M \in H^2(X;\mathbb{Z})$.

<u>Pf</u>: Let $f,g : X \to \mathbb{C}P^\infty$ classify L and M (respectively). Then by the Claim, $\mu\circ(f,g)$ classifies $L\otimes M$, so we just need to show that $(\mu\circ(f,g))^*(\alpha) = f^*\alpha + g^*\alpha$ (where $\alpha \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ is the canonical generator, i.e. $\alpha = c_1(\gamma_1)$).

By the Kunneth Theorem, we can write $\mu^*\alpha = \sum \beta_i \otimes \gamma_i$, where $\deg(\beta_i) + \deg(\gamma_i) = \deg(\mu^*\alpha) = 2$. So since $H^1(\mathbb{C}P^\infty;\mathbb{Z}) = 0$, we in fact can write $\mu^*\alpha = \beta\otimes 1 + 1\otimes\gamma$, with $\beta,\gamma \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$. We claim that $\beta = \gamma = \alpha$: this is where we use Part 1 of the Claim on p. 3.

Letting $i_1 : \mathbb{C}P^\infty \hookrightarrow \mathbb{C}P \times \mathbb{C}P^\infty$ be the first inclusion, we have

$$i_1^*(\beta\otimes 1 + 1\otimes\gamma) = i_1^*(\beta\otimes 1) + i_1^*(1\otimes\gamma) \overset{\text{Kunneth isom}}{=} i_1^*(\pi_1^*\beta \cup \pi_2^*(1)) + i_1^*(\pi_1^* 1 \cup \pi_2^*\gamma)$$
$$= (i_1^*\pi_1^*\beta)\cup 1 + 1 \cup (i_1^*\pi_2^*\gamma) = \beta.$$

But $\mu\circ i_1 \simeq \mathrm{Id}_{\mathbb{C}P^\infty}$, so $i_1^*(\beta\otimes 1 + 1\otimes\beta) = i_1^*\mu^*(\alpha) = \alpha$. So $\beta = \alpha$, and similarly $\gamma = \alpha$. $\square$

Now $c_1(L\otimes M) = (f,g)^*\mu^*\alpha = (f,g)^*(\alpha\otimes 1 + 1\otimes\alpha) = f^*\alpha + g^*\alpha$ as desired. $\square$