Math 601 - Vector Bundles Homework III

Please turn in solutions to at least 3 problems by May 5.

Problem 1 (K-theory of $\mathrm{R}P^2$) Use the long exact sequence in (reduced) K-theory to compute $\widetilde{K}^0(\mathrm{R}P^2)$ and $K^1(\mathrm{R}P^2) := \widetilde{K}^0(S(\mathrm{R}P^2))$. (Hint: think about building $\mathrm{R}P^2$ as a CW complex, and use the long exact sequence obtained from the pair $(\mathrm{R}P^2, \mathrm{R}P^2 - \{*\})$, where $*$ is the center of the single 2-cell.)

Problem 2 (determinants and the first Chern class) Prove that for any complex vector bundle E (over a paracompact base space), we have $c_1(E) = c_1(\det E)$. (Hint: use the Splitting Principle; that is, begin with the case where E is a direct sum of line bundles. Paracompactness is just needed to ensure that the Splitting Principle applies.)

Problem 3 (Vector Bundles and Projective Modules) Given a (real) vector bundle $E \xrightarrow{\pi} X$, show that vector space of sections $\Gamma(E) = \{s : X \to E \,|\, \pi \circ s = \mathrm{Id}_X\}$ forms a module over the ring $C(X)$ of continuous functions from $X \to \mathrm{R}$, and show that if X is compact Hausdorff, then $\Gamma(E)$ is a *projective* module over $C(X)$. Remember that a module M over a ring R is projective if it is a direct summand of a free module. (Hint: see Milnor-Stasheff problem 3-F, and Proposition 1.4 in Hatcher's Vector Bundles notes.)

Problem 4 ($\mathbb{C}P^\infty$ is homotopy commutative) In class we studied the "multiplication" map $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ (we defined μ to be a classifying map for the line bundle $\pi_1^*\gamma^1_\infty \otimes \pi_2^*\gamma^1_\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$; see the notes for Lectures 23-25). Prove that μ is a homotopy commutative multiplication. That is; show that $\mu \circ \tau \simeq \mu$, where $\tau : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ is the "twist" map $\tau(x, y) = (y, x)$. (Hint: this is similar to the proof that μ has an inverse up to homotopy, which was proven in class and in the notes.)

Problem 5 (Characteristic classes of tensor products) Given 2-dimensional complex vector bundles E and F over a space X, compute the characteristic classes $c_1(E \otimes F)$ and $c_2(E \otimes F)$ in terms of the classes $c_1(E), c_2(E), c_1(F)$, and $c_2(F)$. (Hint: you can do this using the Splitting Principle, but it's more fun – and quicker – to use multiplicativity of the Chern Character: $ch(E \otimes F) = ch(E) \cup ch(F)$. It's not much harder to compute $c_3(E \otimes F)$ and $c_4(E \otimes F)$, but the formulas get very large.)