Math 601 - Vector Bundles Homework II
Due March 31

Problem 1 (Dual Bundles)
In this problem, $\mathbb{F}$ will denote either the field $\mathbb{R}$ of real number, or the field $\mathbb{C}$ of complex numbers. Given a vector bundle $E \to B$ with fiber $\mathbb{F}^n$, the dual bundle $E^* = \mathrm{Hom}_F(E, \mathbb{F})$ is defined using the construction in MS Chapter 3.

a) Say $\phi_i : U_i \times \mathbb{F}^n \to E|_{U_i}$ are trivializations with $U_1 \cap U_2 \neq \emptyset$. Describe the transition function for E^* in terms of the transition function $\phi_2^{-1}\phi_1$.

Remark 0.1. *You'll need to express the dual space as a* covariant functor *from vector spaces over* $\mathbb{F}$ *(and isomorphisms) to vector spaces over* $\mathbb{F}$ *(and isomorphisms), so that the construction in MS Chapter 3 applies. Also, it's important to notice that in MS the trivializations of* E^* *come from* $(\mathbb{F}^n)^*$*, rather than from* F^n*, so you'll need to compose with the natural isomorphism between* $\mathbb{F}^n$ *and* $(\mathbb{F}^n)^*$*.*

b) Consider the natural embedding $\mathbb{F}P^1 \to \mathbb{F}P^2$ given by sending $[x, y] \in \mathbb{F}P^1 = (F^2 - \{0\})/F^\times$ to $[x, y, 0] \in \mathbb{F}P^2 = (F^3 - \{0\})/F^\times$. There is a natural projection

$$\pi : \mathbb{F}P^2 - \{[0, 0, 1]\} \longrightarrow \mathbb{F}P^1$$

given by $[x, y, z] \mapsto [x, y, 0]$ (where we identify $\mathbb{F}P^1$ with its image under the embedding), and each fiber of this projection is just a copy of F. Show that this projection is locally trivial over the open sets $\{[x, y, 0] \in \mathbb{F}P^1 : x \neq 0\}$ and $\{[x, y, 0] \in \mathbb{F}P^1 : y \neq 0\}$ (hence π is a vector bundle), and compute the transition function defined by the two trivializations. (This bundle over $\mathbb{F}P^1$ is the *normal bundle* of $\mathbb{F}P^1$ in $\mathbb{F}P^2$.)

c) Recall that the tautological bundle over $\mathbb{F}P^1$ can be written as

$$\{[x, y], \vec{\mathbf{v}} \in \mathbb{F}P^1 \times F^2 : \vec{\mathbf{v}} = (tx, ty) \text{ for some } t \in \mathbb{F}\}$$

By comparing the transition functions you found in b) with transition functions for the tautological bundle over $\mathbb{F}P^1$, show that when $\mathbb{F} = \mathbb{R}$, the normal bundle is isomorphic to the tautological bundle γ_1^1 over $\mathbb{RP}^1 \cong S^1$ (the "Mobius" bundle), and that when $\mathbb{F} = \mathbb{C}$, the normal bundle is the *dual* of the tautological bundle over $\mathbb{C}P^1 \cong S^2$.

Remark 0.2. *In the real case, any metric on a bundle* E *determines an isomorphism* $E \cong E^*$*. In the complex case, this is* not *true, because a Hermitian metric is* conjugate linear *in the second coordinate. In the complex case, one gets an isomorphism* $E^* \cong \overline{E}$*, where* $\overline{E}$ *is the bundle with the same underlying total space as* E*, but with the opposite complex structure: in the bundle* $\overline{E}$*, multiplication by* $z \in \mathbb{C}$ *acts the way multiplication by* $\overline{z}$ *acts on* E*. The bundles* E *and* $E^* \cong \overline{E}$ *are not isomorphic in general, and in particular* $c_1(\overline{E}) = -c_1(E)$*. See MS Chapter 14.*

Problem 2 (Homotopy groups of classifying spaces)
a) Recall that given a group G, the universal principal G–bundle is a principal G–bundle

$$EG \longrightarrow BG$$

for which $\pi_*(EG) = 0$ for $* = 0, 1, \ldots$. Use the long exact sequence in homotopy associated to the fibration $EG \to BG$ to describe the relationship between $\pi_* BG$ and $\pi_* G$.

b) Show that $\mathrm{GL}_n(\mathbb{C})$ and $U(n)$ are connected. What can you conclude about $\pi_1 \mathrm{Gr}_n(\mathbb{C}^\infty)$, and what does this say about complex bundles over 1-dimensional CW complexes? (You may use the fact that the Gram-Schmidt process gives a deformation retraction from $\mathrm{GL}_n(\mathbb{C})$ to $U(n)$.)

c) The action of the unitary group $U(n)$ on $\mathbb{C}^n$ preserves lengths, and hence $U(n)$ acts on $S^{2n-1} \subset \mathbb{C}^n$. Show that this action is transitive, and that the stabilizer of a point under this action is homeomorphic $U(n-1)$. This gives a homeomorphism $U(n)/U(n-1) \cong S^{2n-1}$, and it is a general fact that quotient maps for Lie groups are principal bundles. So we have a principal bundle

$$\begin{array}{ccc} U(n-1) & \longrightarrow & U(n) \\ & & \downarrow \\ & & S^{2n-1}. \end{array}$$

Use this to show that $\pi_1 U(n) \cong \mathbb{Z}$, and $\pi_2 U(n) \cong 0$, and then use part a) to calculate $\pi_2 \mathrm{Gr}_n(\mathbb{C}^\infty)$ and $\pi_3 \mathrm{Gr}_n(\mathbb{C}^\infty)$.