Math 601 - Vector Bundles Homework I

Please read all of the problems, since the statements are all important. Turn in solutions to 4 of the following 5 problems by Monday, February 15th.

Problem 1 (Smooth maps)

Let $M^d \subset \mathbb{R}^n$ be a d–manifold in the sense defined in class (or in Milnor–Stasheff), and let $V \subset \mathbb{R}^m$ be an open subset of some Euclidean space. Show that the following two conditions on a function $f : V \to M^d$ are equivalent. (Functions satisfying these two equivalent conditions are called *smooth*.)

- The composition $i \circ f\colon \mathbb{R}^m \to \mathbb{R}^n$ is smooth, where $i : M^d \hookrightarrow \mathbb{R}^n$ is the inclusion.
- For each chart $h\colon U \to M^d$, the composition $h^{-1}f\colon V \to U$ is smooth.

Hint: For the harder direction, you'll need to use the Inverse Function Theorem, which states that if $f : W_1 \to W_2$ is a smooth function between open sets in $\mathbb{R}^n$, and its Jacobian $D_x f$ is invertible at some point $x \in W_1$, then there are open sets $W_1' \subset W_1$ containing x and $W_2' \subset W_2$ containing $f(x)$, and a smooth map $g : W_2' \to W_1'$ such that g is inverse to (the restriction of) f.

Remark: This problem gives a well-defined notion of smooth maps $f : M^d \to N^k$ between manifolds: for each chart $h : U \to M^d$, we require that $f \circ h : U \to N^k$ is smooth in the above sense.

Problem 2 (Projective spaces, see MS Problem 1-B part b)

A) Given a vector $\overrightarrow{\mathbf{x}} = (x_1, \ldots, x_{k+1}) \in S^k$, let $\alpha(\overrightarrow{\mathbf{x}})$ be the $(k+1) \times (k+1)$–matrix with (i,j)th entry $x_i x_j$. Show that the mapping

$$\alpha : S^k \longrightarrow M_{k+1}(\mathbb{R})$$

induces a homeomorphism from $\mathbb{R}P^k$ to the subspace

$$P^k = \{A \in M_{k+1}(\mathbb{R}) \,|\, A^T = A,\ AA = A,\ \text{and } \mathrm{trace}(A) = 1\}.$$

B) Show that the subspace $P^k \subset M_{k+1}(\mathbb{R}) \cong \mathbb{R}^{(k+1)^2}$ is a smooth manifold of dimension k. (This now gives $\mathbb{R}P^k$ the structure of a smooth manifold.)

C) The tangent bundle to S^k has an action of the group $\mathbb{Z}/2$, given by sending $(x, \alpha'(0))$ to $(-x, -\alpha'(0))$ (where $x \in S^k$ and α is a smooth curve in S^k with $\alpha(0) = x$). Show that α is a smooth map, and that $D\alpha : TS^k \to TP^k$ induces a homeomorphism from $TS^k/\mathbb{Z}/2$ to TP^k.

Problem 3 (Principal Bundles)

A) Show that if $P \xrightarrow{q} B$ is a principal G–bundle, then there is a well-defined, continuous action of G on P, and q induces homeomorphism $P/G \xrightarrow{\cong} B$.

B) Let X be a topological space and G a topological group. Say G acts continuously on X (that is, the map $X \times G \to X$ is continuous). Show that the quotient map $X \xrightarrow{q} X/G$ has the structure of a principal G–bundle if and only if the following three conditions are satisfied:

(1) The action is free, meaning that if $x \cdot g = x$ for some $x \in X$, and some $g \in G$, then $g = 1$.
(2) There exists an open cover $\{U_i\}_{i \in I}$ of X/G and continuous functions $s_i : U_i \to X$ such that $q \circ s_i = \mathrm{Id}$ (we say that the s_i are *local sections*, or *slices*).
(3) The translation map $t : X \times_{X/G} X \to G$, defined by setting $t(x, y)$ to be the unique $g \in G$ such that $g \cdot x = y$, is a continuous map.

C) Using part b), show that the for any vector bundle $E \to B$, the frame bundle $\mathrm{Fr}(E) \to B$ is a principal $\mathrm{GL}_n(\mathbb{R})$–bundle. (This is a little bit confusing, because strictly speaking, b) refers to the quotient map $\mathrm{Fr}(E) \to \mathrm{Fr}(E)/\mathrm{GL}_n(\mathbb{R})$. So you'll need to check that this quotient space is homeomorphic to B.)

Problem 4 (Mixing)

A) Show that if $P \to B$ is a prinicipal $\mathrm{GL}_n(\mathbb{R})$–bundle, then the mixed bundle $P \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n$ has the structure of a vector bundle (i.e. there are vector space structures on the fibers, and local trivializations).

B) Show that the operations of forming frame bundles and mixing are inverse to one another (up to isomorphism). In other words, show that if $P \to B$ is a principal $GL_n(\mathbb{R})$–bundle and $E \to B$ is a vector bundle, then

$$\mathrm{Fr}\left(P \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n\right) \cong P \quad \text{and} \quad (\mathrm{Fr}(E)) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n \cong E.$$

Problem 5 (Metrics)

A) (MS Problem 2-C) Using a partition of unity, show that every vector bundle over a paracompact space can be given a Euclidean metric.

B) Use the Bundle Homotopy Theorem to show that if $\langle\, , \rangle$ and $\langle\, , \rangle'$ are two metrics on the same vector bundle $E \to B$, then there is a bundle isomorphism $\phi : E \to E$ such that $\langle \phi(\vec{\mathbf{v}}), \phi(\vec{\mathbf{w}}) \rangle' = \langle \vec{\mathbf{v}}, \vec{\mathbf{w}} \rangle$ for all $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in E$.

C) Show that if $P \to B$ is a principal $O(n)$–bundle, then the mixed bundle $P \times_{O(n)} \mathbb{R}^n \to B$ is a Euclidean vector bundle (in other words, show that there is a natural way to put a metric on this bundle).

D) EXTRA CREDIT: Give a direct construction of a map ϕ satisfying part b). See MS Problem 2-E for a description of how to do this.

Remark: Given a Euclidean bundle, one can form the prinicpal $O(n)$–bundle of *orthonormal frames.* Then the processes of mixing and taking orthonormal frame bundles become inverses, just like in Problem 4b).