STABLE REPRESENTATION THEORY
OF INFINITE DISCRETE GROUPS

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

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Abstract

The goal of this thesis is to study representations of infinite discrete groups from a homotopical viewpoint. Our main tool and object of study is Carlsson's deformation K-theory, which provides a homotopy theoretical analogue of the classical representation ring. Deformation K-theory is a contravariant functor from discrete groups to connective Ω-spectra, and we begin by discussing a simple model for the zeroth space of this spectrum. We then investigate two related phenomena regarding deformation K-theory: Atiyah-Segal theorems, which relate the deformation K-theory of a group to the complex K-theory of its classifying space, and excision, which relates the deformation K-theory of an amalgamation to the deformation K-theory of its factors. In particular, we use Morse theory for the Yang-Mills functional to prove an Atiyah-Segal theorem for fundamental groups of compact, aspherical surfaces, and we prove that deformation K-theory is excisive on all free products. Combined with work of Tyler Lawson, the former result yields homotopical information about the stable coarse moduli space of surface-group representations.

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Chapter 1

Introduction

Associated to any discrete group Γ, one has the (unitary) representation ring $R(\Gamma)$, which consists of "virtual isomorphism classes" of representations. To form $R(\Gamma)$, one starts with the *sets* $\mathrm{Hom}(\Gamma, U(n))$, and kills the conjugation action of $U(n)$. Block sum of unitary matrices makes the collection of these sets, as n varies, into an abelian monoid, and $R(\Gamma)$ is the Grothendieck group of this monoid. Hence $R(\Gamma)$ consists of formal differences between (isomorphism classes of) representations.

This process completely ignores the fact that the sets $\mathrm{Hom}(\Gamma, U(n))$ have a natural topology, coming from the topology on the groups of unitary matrices. To be precise, one may take the compact-open topology, or (equivalently) the topology coming from the embedding $\mathrm{Hom}(\Gamma, U(n)) \hookrightarrow U(n)^S$, where $S \subset \Gamma$ is any generating set. For finite groups Γ, the space $\mathrm{Hom}(\Gamma, U(n))$ is, topologically speaking, easily understood. The trace of a representation gives a continuous, complete invariant of the isomorphism type, and the trace can take on only countably many values. Hence two non-isomorphic representations are never connected by a path, and since $U(n)$ is connected any two isomorphic representations ρ and $A\rho A^{-1}$ are connected by a path. Hence the component of the representation space containing a given representation ρ is simply the orbit $U(n)/\mathrm{Stab}(\rho)$, and basic representation theory shows that $\mathrm{Stab}(\rho)$ is a product of smaller unitary groups (whose dimensions record the degree with which each irreducible appears in ρ). In particular, when Γ is finite the space $\mathrm{Hom}(\Gamma, U(n))$ depends, topologically speaking, only on the number and dimension of the irreducible representations of Γ.

In contrast, it is well-known that for any Riemann surface M^g, the representation spaces $\mathrm{Hom}(\pi_1(M^g), U(n))$ are connected (see Corollary 4.3.8) and carry a great deal of information. Thus one is inclined to look for an analogue of the representation ring which captures the topology of the representation spaces. As we will discuss, one specific motivation is the desire to prove Atiyah-Segal theorems relating representations of an infinite discrete group Γ to the complex K-theory of the classifying space $B\Gamma$.

Carlsson's deformation K-theory spectrum, first introduced in [10], is precisely the sort of object we want. Its construction may be viewed as the homotopy theoretical analogue of the construction of the representation ring. To form the deformation K-theory spectrum of a discrete group Γ, we replace each step in the construction of $R(\Gamma)$ by its homotopy theoretical analogue. We begin by taking the *spaces* $\mathrm{Hom}(\Gamma, U(n))$, and rather than modding out conjugation, we form the homotopy quotients, or homotopy orbit spaces,

$$\mathrm{Hom}(\Gamma, U(n))_{hU(n)} := EU(n) \times_{U(n)} \mathrm{Hom}(\Gamma, U(n))$$

(Here $EU(n)$ denotes the total space of a universal, principal $U(n)$-bundle. We will frequently use the notation X_{hG} for $EG \times_G X$, where G is a topological group acting on a space X.) These homotopy orbit spaces form a *topological* monoid $\mathrm{Rep}(\Gamma)_{hU}$ under block sum, and the deformation K-theory of Γ is the homotopy group completion of this monoid. More precisely, as we explain below, $\mathrm{Rep}(\Gamma)_{hU}$ is the classifying space of a topological permutative category, and deformation K-theory is the associated K-theory spectrum. (This version of deformation K-theory was first described in Lawson's thesis [29].)

The first two homotopy groups of $K_{\mathrm{def}}(\Gamma)$ have rather direct meanings. In dimension zero, the group $K^0_{\mathrm{def}}(\Gamma)$ is the group of "virtual path components" of representations, i.e. formal differences $[\rho_1]-[\rho_2]$, where $\rho_i \in \mathrm{Hom}(\Gamma, U(n_i))$ for some n_i and square brackets denote path components. This elementary fact is proven in Lemma 2.0.5. The group $K^1_{\mathrm{def}}(\Gamma)$ is essentially a version of $\pi_1\left(\mathrm{Hom}(\Gamma, U(n))/U(n)\right)$, stabilized with respect to rank. A precise result along these lines is proven in Proposition 4.6.1, using a theorem of Lawson regarding the Bott map in deformation

K-theory [30].

Thus far, there have been relatively few computations of the groups $K^*_{\text{def}}(\Gamma)$ for Γ an infinite discrete group. Lawson [30] has provided computations for free groups, as well as a product formula [31] computing $K_{\text{def}}(\Gamma_1 \times \Gamma_2)$ in terms of $K_{\text{def}}(\Gamma_1)$ and $K_{\text{def}}(\Gamma_2)$, as modules over the connective K-theory spectrum **ku**. Specifically, Lawson has shown that there is a weak equivalence of **ku**-modules

$$K_{\text{def}}(\Gamma_1 \times \Gamma_2) \simeq K_{\text{def}}(\Gamma_1) \underset{ku}{\wedge} K_{\text{def}}(\Gamma_2). \tag{1.1}$$

The results of this thesis add to the list of computations: Theorem 4.4.1 provides a complete calculation of $K^*_{\text{def}}(\pi_1(\Sigma))$, for any compact, aspherical surface Σ, and Theorem 4.7.5 computes the deformation K-theory of a free product $\Gamma_1 * \Gamma_2$ in terms of the deformation K-theory of Γ_1 and Γ_2.

The results of this thesis focus on three topics in deformation K-theory: group completion (Chapter 3), which provides, under suitable conditions, convenient models for the zeroth space of the deformation K-theory spectrum; Atiyah-Segal theorems (Chapter 4), which relate $K_{\text{def}}(\Gamma)$ to complex K-theory of the classifying space $B\Gamma$; and excision (Chapter 5), which studies the behavior of deformation K-theory on amalgamated products of groups. We proceed to explain these topics in greater detail.

The Group Completion Theorem [8, 16, 33] provides a homological model for the group completion ΩBM of a topological monoid M. In Chapter 3, we provide conditions (Theorem 3.0.11) under which this homological model actually has the same (weak) homotopy type as the group completion, rather than just the same homology. Furthermore, these conditions are satisfied quite generally for deformation K-theory, and this provides us with a model for the homotopy type of the zeroth space of the spectrum $K_{\text{def}}(\Gamma)$ (Corollary 3.0.16). This model is crucial for the excision results in Chapter 5. We also use this result (or rather special cases of it) in Chapter 4, both as the starting point for our Atiyah-Segal theorem, and in our results on the stable coarse moduli space of representations.

The classical theorem of Atiyah and Segal [7] states that for a compact Lie group Γ, the complex K-theory of the classifying space $B\Gamma$ is isomorphic to the completion of the representation ring $R(\Gamma)$ (at the augmentation ideal). A central goal of this thesis is to provide an analogue of this result, relating the deformation K-theory of $\pi_1\Sigma$, where Σ is a compact, aspherical surface, to the complex K-theory of Σ itself. Since Σ is aspherical, we have $\Sigma = B\left(\pi_1\Sigma\right)$, so this is indeed an analogue of the Atiyah-Segal theorem. More precisely, we prove in Theorem 4.4.1 that there is an isomorphism

$$K^*_{\mathrm{def}}(\pi_1(\Sigma)) \cong K^*(\Sigma);$$

in the orientable case, we require $* > 0$. (We note that Lawson's product formula (1.1) provides an alternate proof in the genus 1 case, since in this case $\pi_1\Sigma = \mathbb{Z}\times\mathbb{Z}$.) Using similar methods, we also study the representation spaces themselves (Section 4.5), obtaining in particular the homotopy type of the stable representation space $\mathrm{Hom}(\pi_1\Sigma, U)$ and the connectivity of the inclusions

$$\mathrm{Hom}(\pi_1\Sigma, U(n)) \hookrightarrow \mathrm{Hom}(\pi_1\Sigma, U(n+1)).$$

Furthermore, combining our results with Lawson's work on the Bott map in deformation K-theory [30], we obtain results regarding the "stable coarse moduli space" $\mathrm{Hom}(\pi_1\Sigma, U)/U$ (Section 4.6).

The proofs of these results rely on Morse theory for the Yang-Mills functional, as developed by Atiyah and Bott [6], Daskalopoulos [12], and Råde [40]. (The key analytical input comes from Uhlenbeck's compactness theorem [47, 48].) The link between deformation K-theory and Yang-Mills theory is provided by the well-known fact that representations of the fundamental group induce flat connections, which form a critical set for the Yang-Mills functional.

In Chapter 5, we discuss the question of excision in deformation K-theory. Given an amalgamated product of groups, one may apply deformation K-theory to obtain a square of spectra, and we say that deformation K-theory is excisive on the amalgamated product if this diagram of spectra is homotopy cartesian. Our main result, Theorem 5.1.1, shows that deformation K-theory satisfies excision for all *free* products. The proof depends crucially on the group completion results from

Chapter 3.

For more general amalgamated products, excision may fail in low dimensions. In Section 5.1, we show that the fundamental group of a Riemann surface, described as an amalgamated product via a connected sum decomposition of the surface, fails to satisfy excision on π_0 (we expect that the natural map ϕ induces an isomorphism in positive degrees, though; see Conjecture 5.0.7). In Section 5.2, we offer several results regarding more general amalgamated products: we show how to deduce excision results in deformation K-theory from stable information about the representation spaces themselves (Proposition 5.2.4), and we study excision in low dimensions for some specific examples of amalgamated products (Proposition 5.2.5). We conclude Section 5.2 by describing a technique, involving stratified fibrations, which we hope will be useful in further work on excision.

It is interesting to note that excision and Atiyah-Segal theorems are closely related phenomena. This is due to the fact that complex K-theory is a cohomology theory, and in particular satisfies excision. More precisely, consider an amalgamated product $G *_K H$ in which the maps $K \to G$ and $K \to H$ are injective. Then the classifying space $B(G *_K H)$ is the homotopy pushout of the diagram

$$BG \longleftarrow BK \longrightarrow BH,$$

and hence one has a long-exact Mayer-Vietoris sequence in complex K-theory.

Given Atiyah-Segal theorems relating deformation K-theory of the factors G, K, and H to the K-theory of their classifying spaces, one then expects that an Atiyah-Segal theorem for $G *_K H$ will be equivalent to excision (in deformation K-theory) for this amalgamated product. More precisely, if the square of deformation K-theory spectra associated to $G *_K H$ is homotopy cartesian, then one has a Mayer-Vietoris sequence in deformation K-theory as well as in topological K-theory, and an isomorphism between $K^*_{\text{def}}(G *_K H)$ and $K^*(B(G *_K H))$ should follow from the 5-lemma; on the other hand, if one has an Atiyah-Segal theorem relating $K^*_{\text{def}}(G *_K H)$ to $K^*(B(G *_K H))$, then the Mayer-Vietoris sequence in complex K-theory should correspond to a Mayer-Vietrois sequence in deformation K-theory, allowing one to prove excision. The difficulty here, as the reader may have guessed, is naturality. In order to transfer information between deformation K-theory and topological K-theory, one needs a natural transformation connecting the functor $K_{\mathrm{def}}(-)$ to the function spectrum $F(B(-), \mathbf{ku})$. Although we describe one such natural transformation in Chapter 2, it is not known to be an equivalence in any interesting cases. In particular, the known Atiyah-Segal theorems involve very different maps, and in interesting cases such as connected sum decompositions of Riemann surfaces, these maps are not natural with respect to the amalgamated product structures.

The final chapter of this thesis focuses on examples. In particular, we consider several families of groups in which the group completion results from Chapter 3 apply, yielding explicit models for the zeroth space of deformation K-theory. In addition, we compute the group K^0_{def} in most of these cases. These results rely heavily on work of Ho and Liu [24, 25], who designed a simple obstruction theory for studying path components of representation spaces and paired it with the theory of quasi-Hamiltonian moment maps [4] in order to compute $\pi_0 \mathrm{Hom}(\pi_1\Sigma, G)$ for (most) surfaces Σ and any compact, connected Lie group G. We specialize their work to the case of the unitary groups, where some simplifications are possible, and then extend their arguments to "surface-type groups," that is, groups with presentations similar to surface groups (Theorem 6.1.9). Proposition 6.1.11 discusses a family of groups related to the Klein bottle, and finitely generated abelian groups are discussed in Proposition 6.2.2.

In an appendix, we discuss the results regarding holonomy of flat connections that are needed in Chapter 4. These results are probably well-known, but no written account seems to be available.

Chapter 2

Deformation K-theory: basic properties

In this chapter, we introduce Carlsson's notion of deformation K-theory and discuss its basic properties. Deformation K-theory is a contravariant functor from discrete groups to spectra, and is meant to capture homotopy-theoretical information about the representation spaces of the group in question. We will construct a connective Ω-spectrum $K_{\text{def}}(G)$ by considering the K-theory of an appropriate permutative topological category of representations (this category was first introduced by Lawson [29]). Although we phrase everything in terms of the unitary groups $U(n)$, all of the constructions, definitions and results in this section are valid for the general linear groups $GL_n(\mathbb{C})$, and only notational changes are needed in the proofs.

For the rest of this section, we fix a discrete group G.

Definition 2.0.1 *Associated to G we have a topological category $\mathcal{R}(G)$ with object space*

$$\mathrm{Ob}(\mathcal{R}(G)) = \coprod_{n=0}^{\infty} \mathrm{Hom}(G, U(n))$$

and morphism space

$$\mathrm{Mor}(\mathcal{R}(G)) = \coprod_{n=0}^{\infty} U(n) \times \mathrm{Hom}(G, U(n)).$$

The domain and codomain maps are $\mathrm{dom}(A, \rho) = \rho$ and $\mathrm{codom}(A, \rho) = A\rho A^{-1}$, and composition is given by $(B, A\rho A^{-1}) \circ (A, \rho) = (BA, \rho)$. The representation spaces are topologized using the compact-open topology, or equivalently as subspaces of $\prod_{g\in S} U(n)$, where $S \subset G$ is any generating set. We define $U(0)$ to be the trivial group, and the single point $* \in \mathrm{Hom}(G, U(0))$ will serve as the basepoint.

The functor $\oplus : \mathcal{R}(G) \times \mathcal{R}(G) \to \mathcal{R}(G)$ defined via block sums of unitary matrices is continuous and strictly associative, with the trivial representation $* \in \mathrm{Hom}(G, U(0))$ as unit, and this functor makes $\mathcal{R}(G)$ into a permutative category in the sense of [32]. The natural commutativity isomorphism

$$c : \rho \oplus \psi \xrightarrow{\cong} \psi \oplus \rho$$

is defined via the (unique) permutation matrices $\tau_{n,m}$ satisfying

$$\tau_{n,m}(A \oplus B)\tau_{n,m}^{-1} = B \oplus A$$

for all $A \in U(n)$ and $B \in U(m)$. (A general discussion of the functor associated to a collection of matrices like this one can be found in the proof of Corollary 3.0.16.) Any homomorphism $f : G \to H$ induces a functor $f^* : \mathcal{R}(H) \to \mathcal{R}(G)$ in the obvious manner, and it is easy to check that this functor is permutative.

May's machine [32] constructs a (special) Γ-category (in the sense of [42]) associated to any permutative (topological) category $\mathcal{C}$. Taking geometric realizations yields a special Γ-space, and Segal's machine then produces a connective Ω-spectrum $K(\mathcal{C})$, the K-theory of the permutative category $\mathcal{C}$. This entire process is functorial in the permutative category $\mathcal{C}$.

Definition 2.0.2 *Given a discrete group G, the deformation K-theory spectrum* *of G is defined to be the K-theory spectrum of the permutative category $\mathcal{R}(G)$, i.e.*

$$K_{\text{def}}(G) = K(\mathcal{R}(G)).$$

This spectrum is contravariantly functorial in G.

We now describe the zeroth space of the spectrum $K_{\text{def}}(G)$.

Lemma 2.0.3 *For any discrete group G, the zeroth space of $K_{\text{def}}(G)$ is naturally weakly equivalent to $\Omega B(|\mathcal{R}(G)|)$, where B denotes the bar construction on the topological monoid $|\mathcal{R}(G)|$.*

The proof of this result is just an elaboration of the proof in [32] that the Γ-category associated to a permutative category $\mathcal{C}$ is special. May constructs levelwise maps [32, Construction 10, Step 2] from $B(|\mathcal{C}|)$ to the first space of the K-theory spectrum of $\mathcal{C}$. One checks that these fit together into a simplicial map, which is a levelwise weak-equivalence. (May also constructs a sequence of maps in the other direction, but they do not form a simplicial map. Nevertheless, levelwise they provide homotopy inverses, showing that our map is a levelwise weak-equivalence.) Since the identity element of $\mathcal{R}(G)$ is disjoint, these simplicial spaces are good and this levelwise equivalence is a weak-equivalence on classifying spaces.

Next, we discuss an observation due to Lawson [29] regarding the classifying space of the category $\mathcal{R}(G)$. For convenience of the reader, and to set notation, we include a discussion of the simplicial constructions of the classifying space $BU(n)$ and the universal bundle $EU(n)$.

Associated to G we have the homotopy orbit spaces

$$\mathrm{Hom}(G, U(n))_{hU(n)} = EU(n) \times_{U(n)} \mathrm{Hom}(G, U(n)),$$

where $EU(n)$ denotes the total space of a universal principal $U(n)$-bundle. In fact, we take $EU(n)$ to be the classifying space of the translation category $\overline{U(n)}$ of $U(n)$, that is, the topological category whose object space is $U(n)$ and whose morphism space is $U(n) \times U(n)$. (The morphism (A, B) is the unique morphism from B to A in $\overline{U(n)}$.) This category admits a right action by $U(n)$ via right multiplication; the induced action on kth level of the nerve of $\overline{U(n)}$ may be written $(A_1, \ldots, A_k) \cdot g = (A_1 g, \ldots A_k g)$. We define $BU(n)$ to be the classifying space of the topological category $\mathcal{C}_{U(n)}$ with one object and with morphism space $U(n)$. The natural functor $\overline{U(n)} \to \mathcal{C}_{U(n)}$ sending the morphism (A, B) to $AB^{-1} \in U(n)$ gives a map $EU(n) \to BU(n)$ making $EU(n)$ a universal principal $U(n)$ bundle (see [41]). The continuous block sum maps $\oplus : U(n) \times U(m) \to U(n+m)$ extend to maps $EU(n) \times EU(m) \to EU(n+m)$, and allow us to define a monoid structure on the disjoint union

$$\coprod_{n=0}^{\infty} \mathrm{Hom}(G, U(n))_{hU(n)} = \coprod_{n=0}^{\infty} EU(n) \times_{U(n)} \mathrm{Hom}(G, U(n)),$$

which we (abusively) denote by $\mathrm{Rep}(G)_{hU}$. Lawson's observation, then, is:

Proposition 2.0.4 (Lawson) *The topological monoids* $|\mathcal{R}(G)|$ *and* $\mathrm{Rep}(G)_{hU}$ *are isomorphic.*

Proof. We begin by considering $\mathrm{Hom}(G, U(n))$ as a constant simplicial space, so that

$$EU(n) \times \mathrm{Hom}(G, U(n)) \cong \left| k \mapsto U(n)^{k+1} \times \mathrm{Hom}(G, U(n)) \right|.$$

Now, combining the level-wise action of $U(n)$ on $EU(n)$ with the conjugation action of $U(n)$ on $\mathrm{Hom}(G, U(n))$ gives the simplicial space on the right a simplicial action of $U(n)$, and we have a homeomorphism

$$EU(n) \times_{U(n)} \mathrm{Hom}(G, U(n)) \cong \left| k \mapsto \left(U(n)^{k+1} \times \mathrm{Hom}(G, U(n)) \right) / U(n) \right|.$$

We will now describe a simplicial map from the right hand side to $N\mathcal{R}(G)$. We will write $N_k\mathcal{R}(G)$, the space of k-tuples of composable morphisms, as

$$\coprod_{n=0}^{\infty} U(n)^k \times \mathrm{Hom}(G, U(n))$$

where $(A_k, \ldots, A_1, \rho)$ is considered as the string of morphisms

$$\rho \xrightarrow{A_1} A_1 \rho A_1^{-1} \xrightarrow{A_2} \ldots \xrightarrow{A_k} A_k \cdots A_1 \rho A_1^{-1} \cdots A_k^{-1}.$$

Then we have a map

$$U(n)^{k+1} \times \mathrm{Hom}(G, U(n)) \longrightarrow U(n)^k \times \mathrm{Hom}(G, U(n)),$$

given by

$$(A_{k+1}, \ldots, A_1, \rho) \mapsto (A_{k+1}A_k^{-1}, A_kA_{k-1}^{-1}, \ldots, A_2A_1^{-1}, A_1\rho A_1^{-1}).$$

It is easy to check that this map is simplicial and factors through the $U(n)$-action on the left, inducing a level-wise homeomorphism from the quotient. Hence we have the desired homeomorphism $\mathrm{Hom}(G, U(n))_{hU} \cong |\mathcal{R}(G)|$, and since both monoid structures arise from block sum, it is immediate from the definitions that this map is a homomorphism of monoids. □

We end this section with a simple observation regarding the zeroth homotopy group of deformation K-theory. The topological monoid $\mathrm{Rep}(G)$ is defined by

$$\mathrm{Rep}(G) = \coprod_{n=0}^{\infty} \mathrm{Hom}(G, U(n)).$$

The monoid structure on $\mathrm{Rep}(G)$ is given by block sum of representations (again, $U(0)$ is the trivial group and the single element in $\mathrm{Hom}(G, U(0))$ will act as the identity). The same construction may be applied with the general linear groups in place of the unitary groups, and we keep the notation intentionally vague.

Lemma 2.0.5 *Let G be a discrete group. Then $K^0_{\mathrm{def}}(G) \cong Gr(\pi_0(\mathrm{Rep}(G)))$, where Gr denotes the group-completion of a monoid, i.e. its Grothendieck group.*

Proof. By Lemma 2.0.3 and Proposition 2.0.4, we know that $K^0_{\mathrm{def}}(G)$ is the group completion of the monoid $\pi_0(\mathrm{Rep}(G))_{hU}$, so we just need to show that there is an isomorphism of monoids $\pi_0(\mathrm{Rep}(G))_{hU} \cong \pi_0(\mathrm{Rep}(G))$. But the monoid

$$\coprod_{n=0}^{\infty} EU(n) \times \mathrm{Hom}(G, U(n))$$

fibers over both sides, with connected fibers. □

We conclude this section by considering the general relationship between deformation K-theory of a group G and complex K-theory of the classifying space BG. Given a natural transformation $K_{\mathrm{def}}(-) \xrightarrow{\eta} \mathrm{Map}(B(-), \mathbf{ku})$ and an amalgamated product $G *_K H$, we may use η, to compare the square of deformation K-theory spectra associated to $G *_K H$ with the square of mapping spectra associated to the classifying spaces of the factors. When the maps from K to G and H are injective, the square of classifying spaces is homotopy co-cartesian [20, p. 92], and hence the square of mapping spectra is homotopy cartesian (i.e. complex K-theory is excisive). Thus if η is an isomorphism on homotopy (in a range) for G, H, and K, then it is an isomorphism (in a range) for $G *_K H$ as well. In the other direction, if η is an isomorphism for $G *_K H$ as well, then excision for complex K-theory implies excision for deformation K-theory. This is relationship, alluded to in the introduction, between Atiyah-Segal theorems and excision.

We briefly describe one such natural transformation η; unfortunately it is not currently known to be an isomorphism on homotopy in any interesting cases (for free groups and surface groups, the isomorphisms arise in completely different manners). The category $\mathcal{R}(G)$ sits inside a larger permutative category $\widetilde{\mathcal{R}}(G)$, whose objects are representations and whose morphisms are all (possibly non-equivariant) linear isomorphisms of the underlying vector spaces. This category has a permutative G-action, which is trivial on objects and sends a morphism $A : \rho \to \psi$ to the morphism $\rho(g)A\psi(g)^{-1} : \rho \to \psi$.

The fixed point category of this action is precisely $\mathcal{R}(G)$; thus $K(\widetilde{\mathcal{R}}(G))^G \cong K_{\mathrm{def}}(G)$. Now $\widetilde{\mathcal{R}}(G)$ is equivalent (as a permutative category) to the full subcategory of trivial representations, on which G acts trivially. Hence the homotopy fixed point spectrum $K(\widetilde{\mathcal{R}}(G))^{hG}$ maps by a weak equivalence to $\mathbf{ku}^{hG} \simeq \mathrm{Map}(BG, \mathbf{ku})$.

Chapter 3

Group completion in deformation K-theory

The goal of this section is to provide a convenient homotopy theoretical model for the group completion of a topological monoid satisfying certain simple properties. The results of this section will be applicable to deformation K-theory, and form the basis of our excision results. In addition, special cases of our main results (Theorem 3.0.11 and Corollary 3.0.16) appear in the computation in Chapter 4 of $K^*_{\mathrm{def}}(\pi_1(\Sigma))$ for compact, aspherical surfaces Σ, and in the related results regarding $\mathrm{Hom}(\pi_1\Sigma, U)/U$.

The models for group completion that we will study arise as mapping telescopes, as in [33]. Throughout this section M will denote a homotopy commutative topological monoid and $e \in M$ will denote the identity element. We write the operation in M as $\oplus$, and for any $m \in M$ we denote the n-fold product of m with itself by m^n.

Definition 3.0.6 *For any $m \in M$, we denote the mapping telescope*

$$\mathrm{hocolim}(\underbrace{M \xrightarrow{\oplus m} M \xrightarrow{\oplus m} \cdots \xrightarrow{\oplus m} M}_{N})$$

by $M_N(m)$, and we denote the infinite mapping telescope

$$\operatorname*{colim}_{N\to\infty} M_N(m) = \operatorname{hocolim}(M \xrightarrow{\oplus m} M \xrightarrow{\oplus m} \cdots)$$

by $M_\infty(m)$.

We denote points in these telescopes by triples (x, n, t), where $x \in M$, $n \in \mathbb{N}$ and $t \in [0, 1)$. Note that each of these spaces is functorial in the pair (M, m) and naturally based by the point $(e, 0, 0)$, which we will denote simply by e.

Definition 3.0.7 *We say that M is* stably group-like *with respect to an element $m \in M$ if the cyclic submonoid of $\pi_0(M)$ generated by m is cofinal. In other words, M is stably group-like with respect to m if for every $x \in M$ there exists $y \in M$ and $n \in \mathbb{N}$ such that $x \oplus y$ and m^n lie in the same path component of M. We refer to such y as* stable homotopy inverses *for x (with respect to m).*

The reason for our terminology is the following result.

Proposition 3.0.8 *Assume M is homotopy commutative and let $m \in M$ be any element. Then there is a natural (abelian) monoid structure on $\pi_0(M_\infty(m))$, and M is stably group-like with respect to m if and only if $\pi_0(M_\infty(m))$ is a group under this multiplication. In fact, if M is stably group-like with respect to m, then $\pi_0(M_\infty)$ is the group completion (i.e. the Grothendieck group) of $\pi_0(M)$.*

Proof. First we describe the monoid structure on $\pi_0(M_\infty(m))$. Given components C_1 and C_2 in $\pi_0(M_\infty(m))$ we may choose representatives $(x_1, n_1, 0)$ and $(x_2, n_2, 0)$ for C_1 and C_2 respectively. Then we define $C_1 \oplus C_2$ to be the component containing $(x_1 \oplus x_2, n_1 + n_2, 0)$. To see that this operation is well-defined, note that if $(x, n, 0)$ and $(x', n', 0)$ are connected by a path, then this path lies in some finite telescope $M_N(m)$ (with $N > n, n'$) and one can collapse the first N mapping cylinders coordinates to obtain a path in M from $x \oplus m^{N-n}$ to $x' \oplus m^{N-n'}$. Hence given any other component, represented by a point $(y, k, 0)$, there is a sequence of paths

$$(x \oplus y, n + k, 0) \sim (x \oplus y \oplus m^{N-n}, N + k, 0) \sim (x \oplus m^{N-n} \oplus y, N + k, 0)$$

$$\sim (x' \oplus m^{N-n'} \oplus y, N+k, 0) \sim (x' \oplus y, n'+k, 0)$$

as desired. This operation is clearly associative and commutative, with the component of $(e, 0, 0)$ as a unit.

Now, say M is stably group-like with respect to $m \in M$. Then given any component C of $M_\infty(m)$, we choose a representative $(x, n, 0)$ for C and a stable homotopy inverse y for x. Then if $x \oplus y \sim m^N$, one easily checks that the component of $(y, N-n, 0)$ is an inverse for C (we may assume, of course, that $N \geqslant n$). Conversely, if $\pi_0(M_\infty(m))$ is a group, then for any $x \in M$ choose a representative $(y, n, 0)$ for the inverse to the component containing $(x, 0, 0)$. Then $(x \oplus y, n, 0)$ lies in the same component as $(m^n, n, 0)$ and hence (by collapsing cylinders) we may construct a path in M from $x \oplus y \oplus m^k$ to m^{n+k} for some k. Thus $y \oplus m^k$ is a stable homotopy inverse for x.

Next we discuss group completions. There is a natural map $\phi : \pi_0(M) \to \pi_0(M_\infty(m))$, given by $\phi([x]) = [x, 0, 0]$ (where square brackets denote the path components containing these points). We must show that a diagram of monoids

$$\begin{tikzcd} \pi_0(M) \arrow[r, "f"] \arrow[dr, "\phi"'] & G \\ & \pi_0(M_\infty(m)) \arrow[u, dotted, "\tilde{f}"'] \end{tikzcd}$$

can be completed uniquely whenever G is a group.

Consider any component $[x, n, t] = [x, n, 0] \in \pi_0(M_\infty(m))$. We may write

$$[x, n, 0] \oplus [m, 0, 0]^n = [x \oplus m^n, n, 0] = [x, 0, 0]$$

and hence we are forced to define

$$\tilde{f}([x, n, 0]) = \tilde{f}([x, 0, 0]) \cdot \tilde{f}([m, 0, 0])^{-n} = f([x]) \cdot f([m])^{-n}. \tag{3.1}$$

It is easy to check that formula (3.1) gives a well-defined function $\tilde{f}$, and it follows from homotopy commutativity that $\tilde{f}$ is a morphism of monoids. □

Example 3.0.9 *If M is homotopy commutative and $\pi_0(M)$ is finitely generated,* *with generators $m_1, \ldots, m_k \in M$, then M is stably group-like with respect to $m = m_1 \oplus \ldots \oplus m_k$: any component is represented by a word in the m_i, and we may add another word to even out the powers. (This example appears, in spirit at least, in [33], and will be central to our results on excision.)*

Before stating the main result of this section, we need the following definition.

Definition 3.0.10 *Let $(M, \oplus)$ be a homotopy commutative monoid. We call an element $m \in M$* anchored *if there exists a homotopy $H : M \times M \times I \to M$ such that for every $m_1, m_2 \in M$, $H_0(m_1, m_2) = m_1 \oplus m_2$, $H_1(m_1, m_2) = m_2 \oplus m_1$, and $H_t(m^n, m^n) = m^{2n}$ for all $t \in I$ and all $n \in \mathbb{N}$.*

Theorem 3.0.11 *Let M be a homotopy commutative monoid which is stably group-like with respect to a anchored element $m \in M$. Then there is a natural isomorphism*

$$\eta : \pi_* M_\infty(m) \xrightarrow{\cong} \pi_* \Omega BM,$$

and the induced map on π_0 is an isomorphism of groups. The map η is induced by a zig-zag of natural weak equivalences.

A number of comments are in order regarding zig-zags, basepoints, and finally, the precise meaning of naturality. First, though, we note that even in the case where M is *strictly* commutative, this result seems non-obvious (although our proof is hardly difficult in this case). This case will be used later, in our calculation of $\pi_1 \mathrm{Hom}(\pi_1 \Sigma, U)/U$ for compact, aspherical surfaces Σ (Section 4.6).

By a zig-zag we simply mean a sequence of spaces $X_1, X_2, \ldots, X_k$, together with maps f_i between X_i and X_{i+1} (in either direction). The sequence of spaces appearing in our natural zig-zag will be made explicit in the proof. A map $f : X \to Y$ between possibly disconnected spaces will be called a weak equivalence if and only if it induces isomorphisms $f_* : \pi_*(X, x) \to \pi_*(Y, f(x))$ for all $x \in X$.

To prove Theorem 3.0.11, we will exhibit a natural zig-zag of weak equivalences between $M_\infty(m)$ and ΩBM. The isomorphism on homotopy groups will then be valid for all compatible choices of basepoint, in the following sense. The zig-zag of isomorphisms on π_0 gives an isomorphism $\eta_0 : \pi_0 M_\infty(m) \to \pi_0 \Omega BM$, and we call basepoints $x \in M_\infty(m)$ and $y \in \Omega BM$ compatible if $\eta_0([x]) = [y]$. Now, for any compatible basepoints x and y there is in fact a canonical isomorphism

$$\eta_{x,y} : \pi_* \left(M_\infty(m), x\right) \xrightarrow{\cong} \pi_* \left(\Omega BM, y\right).$$

This isomorphism is constructed using the fact that if X is a simple space, meaning that the action of $\pi_1(X,x)$ on $\pi_n(X,x)$ is trivial for every $n \geqslant 1$, then any two paths between points $x_1, x_2 \in X$ induce the same isomorphism $\pi_*(X,x_1) \to \pi_*(X,x_2)$). Since we are dealing with a zig-zag of weak equivalences ending with a simple space, all spaces involved are simple, and hence $\eta_{x,y}$ is well-defined.

Naturality means that given a map $f : M \to N$ of monoids with are stably group-like with respect to anchored elements $m \in M$ and $f(m) \in N$, then for any compatible basepoints $x \in M_\infty(m)$ and $y \in \Omega BM$, we have

$$\Omega Bf \circ \eta_{x,y} = f_\infty \circ \eta_{f_\infty(x),(\Omega Bf)(y)} : \pi_* \left(M_\infty(m), x\right) \longrightarrow \pi_* \left(\Omega BN, (\Omega Bf)(y)\right),$$

where $f_\infty = f_\infty(m)$ denotes the map on telescopes induced by f. This equation follows easily from naturality of the weak equivalences involved in the zig-zag.

Remark 3.0.12 *It is possible to relax the definition of "anchored" with out affecting Theorem 3.0.11 (and only minor changes are needed in the proof). For example, the homotopies anchoring m^n need not be the same for all n, and in fact we only need to assume their existence for "enough" n. (In particular, it is not necessary to assume that there is a homotopy anchoring $m^0 = e$.) For all our applications, though, the current definition suffices.*

Before beginning the proof, we discuss the application to deformation K-theory. We work mainly in the unitary case, but all of the results are valid in the general linear case as well (and we have noted the places in which the arguments differ).

For applications to deformation K-theory, our real interests lie in the monoid of homotopy orbit spaces

$$\mathrm{Rep}(G)_{hU} = \coprod_{n=0}^{\infty} \mathrm{Hom}(G, U(n))_{hU(n)},$$

but we can often work with the simpler monoid $\mathrm{Rep}(G)$ instead. Note that the spaces $EU(n)$ are naturally based: recall from Section 2 that $EU(n)$ is the classifying space of a topological category whose object space is $U(n)$; the object $I_n \in U(n)$ provides the desired basepoint $*_n \in EU(n)$. These basepoints behave correctly with respect to block sums, i.e. $*_n \oplus *_m = *_{n+m}$.

Lemma 3.0.13 *Let G be a discrete group. Then $\mathrm{Rep}(G)$ is stably group-like with respect to $\psi \in \mathrm{Hom}(G, U(m))$ if and only if $\mathrm{Rep}(G)_{hU}$ is stably group-like with respect to $[*_m, \psi] \in \mathrm{Hom}(G, U(m))_{hU(m)}$.*

Proof. Say $\mathrm{Rep}(G)$ is stably group-like with respect to $\psi \in \mathrm{Hom}(G, U(m))$. Then given any $[e, \rho] \in \mathrm{Rep}(G)_{hU}$ (with $e \in EU(n)$ and $\rho : G \to U(n)$ for some n) we know that there is a representation $\rho^{-1} : G \to U(k)$ such that $\rho \oplus \rho^{-1}$ lies in the component of the ψ^l (the l-fold block sum of ψ with itself), where $l = \frac{n+k}{m}$. Now, for any $e' \in EU(k)$, the point $[e', \rho^{-1}]$ is a stable homotopy inverse for $[e, \rho]$ (with respect to ψ), since there is a path in $EU(n+k) \times \mathrm{Hom}(G, U(n+k))$ from $[e \oplus e', \rho \oplus \rho^{-1}]$ to $[*_{n+k}, \psi^l]$.

Conversely, if $\mathrm{Rep}(G)_{hU}$ is stably group-like with respect to $[*_m, \psi]$, then any element $[e, \rho] \in \mathrm{Hom}(G, U(n))_{hU(n)}$ has a stable homotopy inverse

$$[e', \rho^{-1}] \in \mathrm{Hom}(G, U(k))_{hU(k)}$$

(for some k), i.e. there is a path in $\mathrm{Hom}(G, U(n+k))_{hU(n+k)}$ from $[e \oplus e', \rho \oplus \rho^{-1}]$ to $[*_{n+k}, \psi^l]$ (where again $l = \frac{n+k}{m}$). Path-lifting for the fibration $EU(n+k) \times \mathrm{Hom}(G, U(n+k)) \to \mathrm{Hom}(G, U(n+k))_{hU(n+k)}$ produces a path in $EU(n+k) \times \mathrm{Hom}(G, U(n+k))$ from $(e \oplus e', \rho \oplus \rho^{-1})$ to some point $(*_{n+k} \cdot A, A^{-1}\psi^l A)$, with $A \in U(n+k)$. The second coordinate of this path, together with connectivity of $U(n)$, shows that $\rho \oplus \rho^{-1}$ lies in the component of ψ^l, i.e. ρ^{-1} is in fact a stable homotopy inverse for ρ (with respect to ψ). $\square$

We will now show that in deformation $\mathrm{Rep}(G)_{hU}$, elements are always anchored. First we need some lemmas regarding the unitary and general linear groups, which are probably well-known.

Lemma 3.0.14 *Consider an element $D = \lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_k I_{n_k} \in GL_n(\mathbb{C})$, where $n = \sum n_i$ and the λ_i are distinct. Then the centralizer of D in $GL_n(\mathbb{C})$ is the subgroup $GL(n_1) \times \cdots \times GL(n_k)$, embedded in the natural manner. As a consequence, the analogous statement holds for the unitary groups.*

Proof. Say $K = [k_{ij}]$ commutes with $X = [x_{ij}]$, where $x_{ij} = 0$ unless $i = j$. Letting e_i denote the ith standard basis vector, the formula $KXe_i = XKe_i$ expands to give $\sum_{j=1}^n k_{ij} x_{ii} e_j = \sum_{j=1}^n k_{ij} x_{jj} e_j$, which precisely states that $k_{ij} = 0$ unless $x_{ii} = x_{jj}$. $\square$

Lemma 3.0.15 *Let $K \subset GL_n(\mathbb{C})$ be any subgroup. Then the set of diagonalizable matrices in the centralizer $C(K)$ is connected. Similarly, for any $K \subset U(n)$, the centralizer of K in $U(n)$ is connected.*

Proof. We prove the general linear case; the argument for $U(n)$ is nearly identical (since by the Spectral Theorem every element of $U(n)$ is diagonalizable).

Let $A \in C(K)$ be diagonalizable. We will produce a path (of diagonalizable matrices) in $C(K)$ from A to the identity. Choose $X \in GL_n(\mathbb{C})$ such that $XAX^{-1} = \lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_k I_{n_k}$ (for some n_i with $\sum n_i = n$). Then by Lemma 3.0.14 we have $XKX^{-1} \subset GL(n_1) \times \cdots \times GL(n_k)$. Now, choose paths $\lambda_i(t)$ from λ_i to 1, lying in $\mathbb{C} - \{0\}$ (or in the unitary case, lying in S^1). This gives a path of matrices Y_t connecting XAX^{-1} to I, and clearly for each $t \in I$ we have $Y_t \in C(XKX^{-1})$. Now $X^{-1} Y_t X$ is a path from A to I lying in $C(K)$. $\square$

Corollary 3.0.16 *Let G be a finitely generated discrete group such that* $\mathrm{Rep}(G)$ *is stably group-like with respect to a representation* $\rho \in \mathrm{Hom}(G, U(k))$. *Then there is a natural isomorphism*

$$\pi_* K_{\mathrm{def}}(G) \cong \pi_* \operatorname{hocolim} \left(\mathrm{Rep}(G,U)_{hU} \xrightarrow{\oplus\rho} \mathrm{Rep}(G,U)_{hU} \xrightarrow{\oplus\rho} \cdots \right),$$

where $\oplus\rho$ denotes block sum with the point $[*_k, \rho] \in \mathrm{Hom}(G, U(k)_{hU(k)}$. *The analogous statement holds for general linear deformation K-theory.*

When $\mathrm{Rep}(G)$ is stably group-like with respect to the trivial representation $1 \in \mathrm{Hom}(G, U(1))$, we will simply say that $\mathrm{Rep}(G)$ is stably group-like.

Remark 3.0.17 *Naturality here has the same meaning as in Theorem 3.0.11, i.e. these spaces are connected by a zig-zag of natural weak equivalences. The comments after Theorem 3.0.11 regarding basepoints apply here as well.*

Examples of groups to which Corollary 3.0.16 applies are provided in Chapters 4 and 6. These examples mainly consist of "surface-like" groups; that is, groups with presentations similar to those for fundamental groups of compact surfaces. In particular, for any compact (possible non-orientable) aspherical surface Σ, the (unitary) representation monoid $\mathrm{Rep}(\pi_1\Sigma)$ is stably group-like with respect to the trivial representation $1 \in \mathrm{Hom}(\pi_1\Sigma, U(1))$ (this result is essentially due to Ho and Liu [25]). We note that there are two approaches to this problem (both originating from work of Ho and Liu), one using Yang-Mills theory and the other using quasi-Hamiltonian moment maps. The former approach is discussed in Corollaries 4.3.8 and 4.3.9, and covers all aspherical surfaces. This approach goes back to [23]. The latter approach is discussed in Theorem 6.1.9 fails for two surfaces (the connected sums of 2 or 4 copies of $\mathbb{R}P^2$), but extends to "surface-like" groups. In the surface case, this approach goes back to [24, 25].

Proof of Corollary 3.0.16. The result will follow immediately from Lemma 2.0.3, Proposition 2.0.4, Lemma 3.0.13, and Theorem 3.0.11 once we show that the element

$$[*_k, \rho] \in \mathrm{Hom}(G, U(k))_{hU(k)}$$

is anchored in the monoid $\mathrm{Rep}(G)_{hU} = |\mathcal{R}(G)|$. We will work with $|\mathcal{R}(G)|$; note that the element $[*_k, \rho]$ above corresponds to the object $\rho \in \mathrm{Hom}(G, U(k))$.

Given any collection of matrices $X = \{X_{(n,m)}\}_{n,m\in\mathbb{N}}$ with $X_{(n,m)} \in U(n+m)$, we can define a functor $F_X : \mathcal{R}(G) \times \mathcal{R}(G) \to \mathcal{R}(G)$ as follows. Given objects $\psi_1 \in \mathrm{Hom}(G, U(n))$ and $\psi_2 \in \mathrm{Hom}(G, U(m))$, we set

$$F_X(\psi_1, \psi_2) = X_{(n,m)}(\psi_1 \oplus \psi_2)X^{-1}_{(n,m)}.$$

We define F_X on morphisms by sending $(A, B) : (\psi_1, \psi_2) \to (A\psi_1A^{-1}, B\psi_2B^{-1})$ to the morphism

$$X_{(n,m)}(\psi_1 \oplus \psi_2)X^{-1}_{(n,m)} \longrightarrow X_{(n,m)}\left((A\psi_1A^{-1}) \oplus (B\psi_2B^{-1})\right)X^{-1}_{(n,m)}$$

represented by the matrix $X_{(n,m)}(A \oplus B)X_{(n,m)}^{-1}$.

Let $\tau_{n,m}$ be the matrix

$$\begin{bmatrix} 0 & I_m \\ I_n & 0 \end{bmatrix}$$

and choose paths $\gamma_{n,m}$ from I_{n+m} to $\tau_{n,m}$ in $U(n+m)$. When $n = m = kl$ ($l \in \mathbb{N}$) we may assume, by Lemma 3.0.15, that $\gamma_{kl,kl}(t) \in \mathrm{Stab}(\rho^{2l})$ for all $t \in I$ (note that $\mathrm{Stab}(\rho^{2l}) = C(\mathrm{Im}\rho^{2l})$). Let X^t denote collection $X^t_{n,m} = \gamma_{n,m}(t)$, and let $F_t = F_{X^t}$ be the associated functor. Then $F_0 = \oplus$ is the functor inducing the monoid structure on $|\mathcal{R}(G)|$, i.e. $|F_0|(x, y) = x \oplus y$, and $|F_1|(x, y) = y \oplus x$. Moreover, at any time t we have $|F_t|(\rho^l, \rho^l) = \rho^{2l}$. The path of functors F_t provides the desired homotopy, proving that $\rho \in |\mathcal{R}(G)|$ is anchored. (Note that a continuous family of functors $G_t : \mathcal{C} \to \mathcal{D}$ defines a continuous functor $\mathcal{C} \times \mathcal{I} \to \mathcal{D}$, where $\mathcal{I}$ denotes the topological category whose object space and morphism space are both the unit interval $[0, 1]$, and hence yields a continuous homotopy). □

Remark 3.0.18 *We note that in the above proof there are obvious natural isomorphisms between F_0 and F_1, given by the matrices $\tau_{n,m}$. This is the usual way to show that a monoid coming from a permutative category is homotopy commutative, but this homotopy does not stabilize ρ.*

We now turn to the proof of Theorem 3.0.11. It will follow easily from the McDuff-Segal Theorem [33] that there is a homology isomorphism

$$H_*(M_\infty(m)) \cong H_*(\Omega BM)$$

with any (abelian) local coefficients. Ordinarily, one would then attempt to show that after applying a plus-construction on the left, these two spaces become weakly equivalent. We will show, though, that the fundamental group of $M_\infty(m)$ is already abelian when m is anchored in M, and hence no plus-construction is required. This will allow us to deduce Theorem 3.0.11.

We begin by showing that all components of $M_\infty(m)$ have abelian fundamental group, and first we discuss the component containing $e = (e, 0, 0)$.

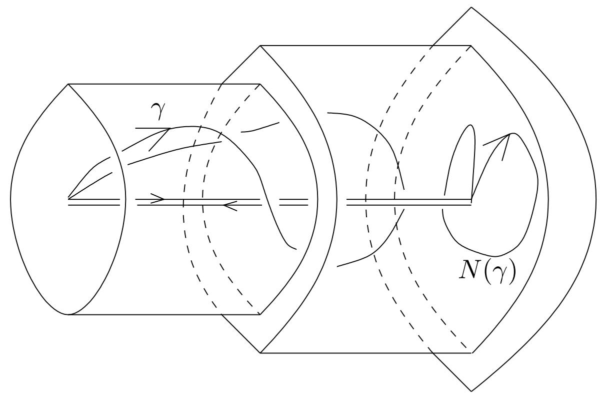

Figure 3.1: A loop γ and its normalization $N(\gamma)$.

Proposition 3.0.19 *Let M be a homotopy commutative monoid. If $m \in M$ is anchored, then $\pi_1(M_\infty(m), e)$ is abelian.*

The idea of the proof is show that the ordinary multiplication in $\pi_1(M_\infty(m), e)$ agrees with an operation defined in terms of the multiplication in M. This latter operation will immediately be commutative, by our assumptions on M.

The proof will require some simple lemmas regarding loops in mapping telescopes. We write $p_1 \centerdot p_2$ for composition of paths (tracing out p_1 first). We begin by describing the type of loops that we will need to use. (Our notation for mapping telescopes was described at the start of this section.)

Definition 3.0.20 *For any n, there is a canonical path $\gamma_n : I \to M_\infty(m)$ starting at the basepoint $e = (e, 0, 0)$ and ending at $(m^n, n, 0)$, defined piecewise by*

$$\gamma_n(t) = \begin{cases} (m^k, k, n(t - k/n)), & \frac{k}{n} \leqslant t < \frac{k+1}{n}, \quad k = 0, \ldots, n-1 \\ \gamma_n(1) = (m^n, n, 0). & \end{cases}$$

We call a loop $\alpha : I \to M_\infty$ normal *(at level n) if it is based at e and has the form $\alpha = \gamma_n \centerdot \widetilde{\alpha} \centerdot \gamma_n^{-1}$, where $\widetilde{\alpha}(I) \subset M \times \{n\} \times \{0\}$. Note that if α is normal, then its "middle third" $\widetilde{\alpha}$ is uniquely defined. We will often think of $\widetilde{\alpha}$ as a loop in M rather than in $M_\infty(m)$.*

Given a normal loop $\gamma_n \centerdot \widetilde{\alpha} \centerdot \gamma_n^{-1}$, we define its kth renormalization to be the normal loop $\gamma_{n+k} \centerdot \widetilde{\alpha} \oplus m^k \centerdot \gamma_{n+k}^{-1}$.

Lemma 3.0.21 *Any loop in $M_\infty(m)$ can be normalized, i.e. is homotopic (rel $\{0,1\}$) to a normal loop. Moreover, any normal loop is homotopic to all of its renormalizations.*

Proof. In general, say we are given a loop γ in the mapping cylinder M_f of a map $f : X \to Y$, and say γ is based at $(x_0, 0) \in M_f$ for some $x_0 \in X$. Then if H_t denotes a homotopy from Id_{M_f} to the retraction $r : M_f \to Y$ (such that $H_t(x_0, 0) = (x_0, t)$ for $t < 1$) we have a homotopy connecting γ to a loop whose middle third lies in Y:

$$\gamma_t(s) = \begin{cases} H_{s(1+2t)}(x_0, 0), & 0 \leqslant s \leqslant \frac{t}{1+2t} \\ H_t\left(\gamma\left((1+2t)s - t\right)\right), & \frac{t}{1+2t} \leqslant s \leqslant \frac{t}{1+2t} \\ H_{1+2t-s(1+2t)}(x_0, 0), & \frac{t}{1+2t} \leqslant s \leqslant 1. \end{cases}$$

Now, any loop in $M_\infty(m)$ lies in some finite telescope $M_N(m)$, and applying the above process N times produces a normal loop (up to reparametrization). Homotopies between a normal loop and its renormalizations are then produced similarly. □

Definition 3.0.22 *Given loops $\widetilde{\alpha}$ and $\widetilde{\beta}$ in M, let $p_{\widetilde{\alpha},\widetilde{\beta}}$ denote their pointwise sum. For normal loops $\alpha = \gamma_n \centerdot \widetilde{\alpha} \centerdot \gamma_n^{-1}$ and $\beta = \gamma_n \centerdot \widetilde{\beta} \centerdot \gamma_n^{-1}$ in $M_\infty(m)$, we define $\alpha \oplus \beta$ to be the normal loop (of level $2n$) given by*

$$\alpha \oplus \beta = \gamma_{2n} \centerdot p_{\widetilde{\alpha},\widetilde{\beta}} \centerdot \gamma_{2n}^{-1}.$$

Lemma 3.0.23 *For any normal loops α and β of level n (in $M_\infty(m)$), there is basepoint preserving homotopy $\alpha \oplus \beta \simeq \beta \oplus \alpha$.*

Proof. Since m is anchored, there is a homotopy $H : M \times M \times I \to M$ such that $H(x, y, 0) = x \oplus y$, $H(x, y, 1) = y \oplus x$, and $H(m^n, m^n, s) = m^{2n}$ for all $s \in I$ and all $n \in \mathbb{N}$. Let $\alpha = \gamma_n \centerdot \widetilde{\alpha} \centerdot \gamma_n^{-1}$, let $\beta = \gamma_n \centerdot \widetilde{\beta} \centerdot \gamma_n^{-1}$, and define $h_s(\alpha, \beta)$ to be the loop

$$h_s(\alpha, \beta)(t) = H(\widetilde{\alpha}(t), \widetilde{\beta}(t), s)$$

(note that $h_s(\alpha, \beta)$ is based at $H(m^n, m^n, s) = m^{2n}$). The family of loops (based at e) given by

$$p_s = \gamma_{2n} \centerdot h_s(\alpha, \beta) \centerdot \gamma_{2n}^{-1}$$

now provides the desired homotopy between $\alpha \oplus \beta$ and $\beta \oplus \alpha$. $\square$

Proof of Proposition 3.0.19. Let a and b be elements of $\pi_1(M_\infty(m), e)$. By Lemma 3.0.21, we may choose normal representatives $\alpha = \gamma_n \centerdot \widetilde{\alpha} \centerdot \gamma_n^{-1}$ and $\beta = \gamma_k \centerdot \widetilde{\beta} \centerdot \gamma_k^{-1}$ for a and b. Renormalizing if necessary, we can assume $k = n$. By Lemma 3.0.23, it suffices to show that $\alpha \centerdot \beta \simeq \alpha \oplus \beta$ (rel $\{0,1\}$). Let m^n denote the constant loop at m^n. Note that the nth renormalization of α is precisely $\alpha \oplus (\gamma_n \centerdot m^n \centerdot \gamma_n^{-1})$ (and similarly for β). Using Lemma 3.0.21 and commutativity of $\oplus$ we now have

$$
\begin{aligned}
\alpha \centerdot \beta &\simeq (\alpha \oplus (\gamma_n \centerdot m^n \centerdot \gamma_n^{-1})) \centerdot (\beta \oplus (\gamma_n \centerdot m^n \centerdot \gamma_n^{-1})) \\
&\simeq (\alpha \oplus (\gamma_n \centerdot m^n \centerdot \gamma_n^{-1})) \centerdot ((\gamma_n \centerdot m^n \centerdot \gamma_n^{-1}) \oplus \beta) \\
&= (\gamma_{2n} \centerdot p_{\widetilde{\alpha}, m^n} \centerdot \gamma_{2n}^{-1}) \centerdot (\gamma_{2n} \centerdot p_{m^n, \widetilde{\beta}} \centerdot \gamma_{2n}^{-1}) \\
&\simeq \gamma_{2n} \centerdot p_{\widetilde{\alpha}, m^n} \centerdot p_{m^n, \widetilde{\beta}} \centerdot \gamma_{2n}^{-1} \\
&= \gamma_{2n} \centerdot p_{\widetilde{\alpha} \centerdot m^n, m^n \centerdot \widetilde{\beta}} \centerdot \gamma_{2n}^{-1} \\
&= (\gamma_n \centerdot (\widetilde{\alpha} \centerdot m^n) \centerdot \gamma_n^{-1}) \oplus (\gamma_n \centerdot (m^n \centerdot \widetilde{\beta}) \centerdot \gamma_n^{-1}).
\end{aligned}
$$

Since $\widetilde{\alpha} \centerdot m^n \simeq \widetilde{\alpha}$, we have a homotopy $\gamma_n \centerdot \widetilde{\alpha}_s \centerdot \gamma_n^{-1}$ from $\gamma_n \centerdot (\widetilde{\alpha} \centerdot m^n) \centerdot \gamma_n^{-1}$ to α (we may assume each loop in this homotopy is normal) and analogously for β. The family of loops $\gamma_n \centerdot \widetilde{\alpha}_s \centerdot \gamma_n^{-1} \oplus \gamma_n \centerdot \widetilde{\beta}_s \centerdot \gamma_n^{-1}$ provides a homotopy from $(\gamma_n \centerdot (\widetilde{\alpha} \centerdot m^n) \centerdot \gamma_n^{-1}) \oplus (\gamma_n \centerdot (m^n \centerdot \widetilde{\beta}) \centerdot \gamma_n^{-1})$ to $\alpha \oplus \beta$, and since all homotopies involved are basepoint preserving, this completes the proof. $\square$

We now show that all components of $M_\infty(m)$ have abelian fundamental group, not just the component containing e.

Corollary 3.0.24 *Let M be a homotopy commutative monoid which is stably group-like with respect to a anchored element $m \in M$. Then all path components of $M_\infty(m)$ have abelian fundamental group.*

Proof. For any element $(x, n, t) \in M_\infty(m)$, we define $C_{(x,n,t)}$ to be the component of $M_\infty(M)$ containing this element. We need to show that $\pi_1(C_{(x,n,t)}(M))$ is abelian. Let $x^{-1} \in M$ be a stable homotopy inverse for x, i.e. an element such that for some

N, $x \oplus x^{-1}$ and m^N lie in the same component of M. Note that by adding m^n to x^{-1} if necessary, we may assume that $N > n$. We will construct maps

$$f : (C_{(x,n,t)}, (x, n, t)) \to (C_{(e,0,0)}, (x^{-1} \oplus x, N, t))$$

and

$$g : (C_{(e,0,0)}, (x^{-1} \oplus x, N, t)) \to (C_{(x,n,t)}, (x \oplus x^{-1} \oplus x, n + N, t))$$

and show that the composition $g_* \circ f_*$ is injective on π_1, from which it follows that f_* is injective. This will suffice, since by Proposition 3.0.19 the group $\pi_1(C_{(e,0,0)}, (x^{-1} \oplus x, N, t))$ is abelian.

The maps f and g are defined by

$$f(y, k, s) = (x^{-1} \oplus y, k + (N - n), s),$$

$$g(y, k, s) = (x \oplus y, k + n, s);$$

note that in both cases these are continuous maps (defined, in fact, on the whole mapping telescope $M_\infty(m)$) and they map the basepoints in the manner indicated above. The composite map is given by $g \circ f(y, k, t) = (x \oplus x^{-1} \oplus y, k + N, t)$.

Consider an element $[\alpha] \in \ker(g_* \circ f_*)$. Then α lies in some finite telescope $M_k(m)$, and by collapsing this telescope to its final stage, we obtain a *free* homotopy from α to a loop $\overline{\alpha}$ lying in $M \times \{k\} \times \{0\}$. Now, we have a free homotopy

$$g \circ f \circ \alpha \simeq g \circ f \circ \overline{\alpha} = x \oplus x^{-1} \oplus \overline{\alpha}$$

where the final loop lies in $M \times \{k + N\} \times \{0\}$. By assumption, there is a path in M from $x \oplus x^{-1}$ to m^N, and together with homotopy commutativity of M we find that $x \oplus x^{-1} \oplus \overline{\alpha} \simeq m^N \oplus \overline{\alpha} \simeq \overline{\alpha} \oplus m^N$. But this loop, lying in $M \times \{k+N\} \times \{0\}$, is clearly homotopic (in $M_\infty(m)$) to the loop $\overline{\alpha}$ lying in $M \times \{k\} \times \{0\}$, which by construction is homotopic to α. Thus we have a free homotopy $g \circ f \circ \alpha \simeq \alpha$, and by assumption $g \circ f \circ \alpha$ is nullhomotopic. Hence α is *freely* nullhomotopic. But freely nullhomotopic loops are always trivial in π_1, so $g_* \circ f_*$ is injective as claimed.
□

Proof of Theorem 3.0.11. To fix notation, we begin by describing the McDuff-Segal approach to group completion [33]. Given a space X together with an (left) action of a monoid M on X, one may form the topological category X_M whose object space is X and whose morphism space is $M \times X$. Here (m, x) is a morphism from x to $m \cdot x$, and composition is given by $(n, mx) \circ (m, x) = (n \oplus m, x)$. There is a natural, continuous functor $Q : X_M \to BM$ where BM denotes the topological category with one object and with morphism space M (the geometric realization of BM is the classifying space of M, which we also denote by BM). On morphisms, this functor sends (m, x) to m. When $X = M$ (acted on via left multiplication) the category M_M has an initial object (the identity $e \in M$) and hence $EM = |M_M|$ is canonically contractible. (Note here that M_M is *not* the category with a unique morphism between any pair of objects). Now, M acts on $M_\infty(m)$ via $x \cdot (y, n, t) = (x \oplus y, n, t)$, and we define $(M_\infty)_M = (M_\infty(m))_M$. This space has a natural basepoint, coming from the basepoint $e \in M_\infty(m)$. Observe that $(M_\infty)_M$ is the infinite mapping telescope of the sequence

$$EM \xrightarrow{F_m} EM \xrightarrow{F_m} \cdots ,$$

where F_m is the functor defined by $F_m(x) = x \oplus m$ and $F_m(n, x) = (n, x \oplus m)$. Since EM is contractible, it follows that $(M_\infty)_M$ is (weakly) contractible as well. Now, as noted above we have a functor $Q_m : (M_\infty)_M \to BM$, and we denote its realization by q_m. The fiber of the map q_m (over the vertex of BM) is precisely $M_\infty(m)$, and so we have a natural map

$$i_m : M_\infty(m) \longrightarrow \mathrm{hofib}(q_M).$$

The theorem of McDuff and Segal [33] states that this map is an isomorphism in homology with local (abelian) coefficients, so long as the action of M on $M_\infty(m)$ is by homology equivalences (again with local coefficients). The hypothesis of this theorem is satisfied when M is stably group-like with respect to m, as follows easily using the fact that homology of a mapping telescope may be computed as a colimit. For completeness, we give a full proof in Lemma 3.0.25 below.

We note that McDuff and Segal actually work with the thick realization $||\cdot||$ of simplicial spaces, meaning that the real conclusion of their theorem is that the map $M_\infty(m) \to \text{hofib}||Q_m||$ is a homology equivalence with local coefficients. (Note that $M_\infty(m)$ is the fiber of both $||Q_m||$ and $q_m = |Q_m|$.) In the present application, the simplicial spaces involved are good, so the thick realization is homotopy equivalent to the ordinary realization [42]. Hence one finds that there is a weak equivalence $\text{hofib}||q|| \to \text{hofib}(q)$, and since weak equivalences induce isomorphisms in homology with local coefficients, we conclude that the map $i_M : M_\infty(m) \to \text{hofib}(q)$ induces isomorphisms in homology with local coefficients as well.

Next, since $(M_\infty)_M$ is (weakly) contractible, we have a weak equivalence from ΩBM to $\text{hofib}(q_M)$, induced by the diagram

$$\begin{tikzcd} * \arrow[r, "\simeq"] \arrow[d] & (M_\infty)_M \arrow[d, "q_m"] \\ BM \arrow[r, equal] & BM. \end{tikzcd}$$

Here the maps from $*$ are the inclusions of the natural basepoints; note that $\Omega BM \cong \text{hofib}(* \to BM)$. Hence we have a natural zig-zag

$$M_\infty(m) \xrightarrow{i_m} \text{hofib}(q_m) \xleftarrow{\simeq} \Omega BM, \tag{3.2}$$

and the first map induces an isomorphism in homology with local (abelian) coefficients. (This is, of course, the full conclusion of the McDuff-Segal Theorem.) By Corollary 3.0.24, all components of $M_\infty(m)$ have abelian fundamental group, and hence i_m induces isomorphisms on $\pi_1 \cong H_1$. It is well-known that a map inducing isomorphisms on homology with local coefficients, and on π_1, is a weak equivalence (see, for example, [20, p. 389, Ex. 12]).

To complete the proof of Theorem 3.0.11, we must show that the zig-zag (3.2) induces an isomorphism of groups $\pi_0(M_\infty(m)) \cong \pi_0(\Omega BM)$ (the multiplication on $\pi_0(M_\infty(m))$ was described in Proposition 3.0.8). We already know that these maps induce a bijection, so it suffices to check that the induced map is a homomorphism. Any component of $M_\infty(m)$ is represented by a point of the form $(x, n, 0)$, with $x \in M$ and $n \in \mathbb{N}$. Now, the fiber of q_m over $* \in BM$ is precisely the objects of the category $(M_\infty)_M$, i.e. the space $M_\infty(m)$, and hence we identify $(x, n, 0)$ with a point in $q_m^{-1}(*)$. Hence we may write $i_m(x,n,0) = ((x,n,0),c_*) \in \mathrm{hofib}(q_m)$, where c_* denotes the constant path at $* \in BM$. Next, recall that since BM is the realization of a category with M as morphisms, every element $y \in M$ determines a loop $\alpha_y \in \Omega BM$. Let ψ denote the natural map from $\Omega BM \to \mathrm{hofib}(q_m)$. We claim the points $\psi\left(\alpha_{m^n}^{-1} \centerdot \alpha_x\right)$ and $((x,n,0),c_*)$ lie in the same path component of $\mathrm{hofib}(q_m)$. This implies that the map $\pi_0(M_\infty(m)) \to \pi_0(\Omega BM)$ sends the component of $(x,n,0)$ to the component of $\alpha_{m^n}^{-1} \centerdot \alpha_x$. Since M is homotopy commutative and $\pi_0(\Omega BM)$ is the group completion of $\pi_0 M$, this map is a homomorphism of monoids.

We now produce the required path between $((x,n,0),c_*)$ and $\psi\left(\alpha_{m^n}^{-1} \centerdot \alpha_x\right)$ (in $\mathrm{hofib}(q_m)$). By definition of the map ψ we have $\psi\left(\alpha_{m^n}^{-1} \centerdot \alpha_x\right) = (e, \alpha_{m^n}^{-1} \centerdot \alpha_x)$, where $e = (e,0,0) \in M_\infty(m)$. There are morphisms in the category $(M_\infty)_M$ from the object $(e,n,0)$ to $(x,n,0)$ and to $(m^n,n,0)$, corresponding (respectively) to the elements x and m^n in M. These morphisms give paths β_x and β_{m^n} in $|(M_\infty)_M|$ which map under q_m to the paths α_x and α_{m^n}, respectively. Letting α_x^t denote the path $\alpha_x^t(s) = \alpha_x(1-t+ts)$, it is easy to check that $(\beta_x(1-t), \alpha_x^t)$ is a path in $\mathrm{hofib}(q_m)$ starting at $((x,n,0),c_*)$ and ending at $((e,n,0),\alpha_x)$. One next constructs an analogous path from $((e,n,0),\alpha_x)$ to $\left((m^n,n,0), \alpha_{m^n}^{-1} \centerdot \alpha_x\right)$. Finally, since $(m^n,n,0)$ and e lie in the same component of $M_\infty(m) = q_m^{-1}(*)$, we have a path in $\mathrm{hofib}(q_m)$ from $\left((m^n,n,0), \alpha_{m^n}^{-1} \cdot \alpha_x\right)$ to $(e, \alpha_{m^n}^{-1} \cdot \alpha_x)$. □

Lemma 3.0.25 *Let M be a homotopy commutative monoid and assume M is stably group-like with respect to $m \in M$. Then the action of M on $M_\infty(m)$ induces isomorphisms in homology with any local (abelian) coefficients.*

Proof. Recall that the action of M on $M_\infty(m)$ is given by $x \cdot (y,n,t) = (x \oplus y, n, t)$. Given $x \in M$, let $f = f_x : M_\infty(m) \to M_\infty(m)$ denote the map induced by this action. We need to show that for any $x \in M$, and for any abelian coefficient system A on $M_m(\infty)$, this map induces an isomorphism

$$H_*(M_\infty(m), f^*(A)) \xrightarrow{f_*} H_*(M_\infty(m), A)$$

in cohomology with local coefficients. Let A_n denote the restriction of A to the finite telescope $M_n(m) \simeq M$ and note that $f^*(A_n) = (f^*A)_n$. Now, under the canonical identifications $M_n(m) \simeq M$ we see that A_n is just the pullback of A_{n+1}

under the map $M \to M$, $x \mapsto x \oplus m$, and similarly for $f^*(A_n)$. Since the homology of the infinite telescope is the colimit of the homology of the finite telescopes, we see that the map f_* is the colimit of the vertical maps in the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\oplus m"] & H_*(M, f^*(A_n)) \arrow[r, "\oplus m"] \arrow[d, "x\oplus"] & H_*(M, f^*(A_{n+1})) \arrow[r, "\oplus m"] \arrow[d, "x\oplus"] & \cdots \\
\cdots \arrow[r, "\oplus m"] & H_*(M, A_n) \arrow[r, "\oplus m"] & H_*(M, A_{n+1}) \arrow[r, "\oplus m"] & \cdots .
\end{tikzcd}$$

Since x has a stable homotopy inverse y with $x \oplus y$ in the connected component of m^k (for some $k > 0$) we have a second diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\oplus m"] & H_*(M, g * f^*(A_n)) \arrow[r, "\oplus m"] \arrow[d, "y\oplus"] & H_*(M, g * f^*(A_{n+1})) \arrow[r, "\oplus m"] \arrow[d, "y\oplus"] & \cdots \\
\cdots \arrow[r, "\oplus m"] & H_*(M, f^*(A_n)) \arrow[r, "\oplus m"] & H_*(M, f^*(A_{n+1})) \arrow[r, "\oplus m"] & \cdots .
\end{tikzcd}$$

The vertical composite of these diagrams is

$$\begin{tikzcd}
\cdots \arrow[r, "\oplus m"] & H_*(M, f^*(A_n)) \arrow[r, "\oplus m"] \arrow[d, "m^k\oplus"] & H_*(M, f^*(A_{n+1})) \arrow[r, "\oplus m"] \arrow[d, "m^k\oplus"] & \cdots \\
\cdots \arrow[r, "\oplus m"] & H_*(M, A_{n+k}) \arrow[r, "\oplus m"] & H_*(M, A_{n+k+1}) \arrow[r, "\oplus m"] & \cdots .
\end{tikzcd}$$

and here the map on colimits is easily seen to be an isomorphism. Repeating the argument with x and y interchanged completes the proof. $\square$

Chapter 4

An Atiyah-Segal theorem for surface groups

A well-known theorem of Atiyah and Segal [7] states that for a compact Lie group Γ, the complex K-theory of the classifying space $B\Gamma$ is isomorphic to the completion of the representation ring $R(\Gamma)$ (at the augmentation ideal). In this chapter, we provide a relationship between (unitary) representations of the fundamental group of a compact aspherical surface M and the K-theory of the surface itself. Of course, when M is aspherical (i.e. when M is neither S^2 nor $\mathbb{R}P^2$), $M = B(\pi_1(M))$.

Our main result (Theorem 4.4.1) shows that the homotopy groups

$$K^*_{\text{def}}(\pi_1(M)) = \pi_* K_{\text{def}}(\pi_1(M))$$

are isomorphic to $K^*(M)$ (in the orientable case, we require $* > 0$). As described in the introduction, one may view deformation K-theory as the homotopical analogue of the representation ring, and hence this is our analogue of the Atiyah-Segal theorem. We note that since the suspension of a surface M breaks up as a wedge, $K^*(M)$ is easily calculated, and hence our main result provides a complete calculation of $K^*_{\text{def}}(\pi_1(M))$. By similar methods, we obtain a number of results (Section 4.5) regarding the topology of the representation spaces themselves. In Theorem 4.5.3, we show that $\mathrm{Hom}(\pi_1(M^g), U)$ is homotopy equivalent to $U^{2g} \times BU$, and we determine the stable range for the inclusions

$$\mathrm{Hom}(\pi_1(M^g), U(n)) \hookrightarrow \mathrm{Hom}(\pi_1(M^g), U(n+1))$$

in most cases. In Section 4.6, we combine our results with work of Lawson in order to study the stable coarse moduli spaces $\mathrm{Hom}(\pi_1 M, U)/U$.

All of these results rely on Morse theory for the Yang-Mills functional. To motivate the arguments, we give a proof, along these lines, of the well-known fact that the free loop space of a connected, compact Lie group G is homotopy equivalent to the homotopy orbit space $EG \times_G G$ (where G acts on itself by conjugation). This result is well-known for *any* group G, but the only reference of which I am aware is the elegant proof given by Gruher in her thesis [18].

To begin, note that $EG \times_G G = \mathrm{Hom}(\mathbb{Z}, G)_{hG}$. Connections A over the circle are always flat, and hence give rise to holonomy representations of $\pi_1 S^1 = \mathbb{Z}$:

$$A \mapsto (\rho_A : \mathbb{Z} \to G)\,.$$

After modding out based gauge transformations (i.e. automorphisms of the principal bundle $S^1 \times G$ which restrict to the identity over $1 \in S^1$), one obtains a homeomorphism (Proposition 4.2.8)

$$\mathcal{A}(S^1 \times G)/\mathrm{Map}_*(S^1, G) \cong \mathrm{Hom}(\mathbb{Z}, G),$$

and since the based gauge group acts freely, a standard fact about homotopy orbit spaces (Lemma 4.2.11) yields a homotopy equivalence

$$\big(\mathcal{A}(S^1 \times G)/\mathrm{Map}_*(S^1, G)\big)_{hG} \simeq \big(\mathcal{A}(S^1 \times G)\big)_{h\mathrm{Map}(S^1, G)}\,.$$

But connections form a contractible (affine) space, so the right hand side is the classifying space of the (full) gauge group. Atiyah and Bott have shown that the space $\mathrm{Map}(S^1, BG) = LBG$ is a model for this classifying space, so we conclude that $EG \times_G G \simeq LBG$, as desired.

Our interest in this argument lies in the fact that deformation K-theory (of $\mathbb{Z}$, say) is built from the homotopy orbit spaces

$$EU(n) \times_{U(n)} \mathrm{Hom}(\mathbb{Z}, U(n)) = EU(n) \times_{U(n)} U(n)$$

(see Proposition 4.1.1), and the homotopy groups of $LBU(n) = \mathrm{Map}(S^1, BU(n))$ are precisely the complex K-groups of $S^1 = B\mathbb{Z}$ (in dimensions $0 < k < 2n$). Thus the statement $EU(n) \times_{U(n)} U(n) \simeq LBU(n)$ may be interpreted as an Atiyah-Segal theorem for the group $\mathbb{Z}$.

When $\mathbb{Z}$ is replaced by the fundamental group of a two-dimensional surface, one can try to mimic this argument. Not all connections are flat in this case, but flat connections do form a critical set for the Yang-Mills functional. Hence one may hope to relate this critical set to the space $\mathcal{A}$ of all connections via the Morse stratification for the Yang-Mills functional, i.e. the stratification of $\mathcal{A}$ by stable manifolds. Råde's work provides deformation retractions from the strata to their critical sets, and in particular allows us to pass from the critical set of flat connections to its stable manifold. By results of Daskalopoulos, this stratification agrees with the Harder-Narasimhan stratification from complex geometry (as was conjectured by Atiyah and Bott) and in particular the stable manifold for the space of flat connections is the space of semi-stable holomorphic structures. We give precise bounds on the codimensions of the Harder-Narasimhan strata, and our main results then follow from an application of Smale's infinite dimensional transversality theorem.

This chapter is organized as follows. In Section 4.1, we explain how to use the group completion results of Chapter 3 to obtain a convenient model for the zeroth space of the Ω-spectrum $K_{\mathrm{def}}(\pi_1 M)$ when M is a compact, aspherical surface. The precise passage from representation varieties to spaces of flat connections, and then to the larger spaces of semi-stable holomorphic structures, is discussed in Section 4.2. In Section 4.3 we discuss the Harder-Narasimhan stratification on the space of holomorphic structures. The main theorem is proven in Section 4.4, using the results of the previous three sections. In Section 4.5, we also study the representation spaces themselves, as the rank tends to infinity, and in Section 4.6

we discuss Lawson's cofiber sequence and its implications for the homotopy groups of the coarse moduli space $\mathrm{Hom}(\pi_1 M, U)/U$. In the final section, we extend our results to free products of surface groups.

4.1 Group completion for surface groups

The starting point for our study of $K_{\mathrm{def}}(\pi_1(M))$ (for M a compact, aspherical surface) is the following result, which gives a convenient model for the zeroth space of this spectrum.

Proposition 4.1.1 *Let M be either the circle or an aspherical compact surface. Then there is a weak equivalence between the zeroth space of $K_{\mathrm{def}}(\pi_1(M))$ and the space*

$$\mathrm{hocolim}\left(\mathrm{Rep}(\pi_1 M)_{hU} \xrightarrow{\oplus 1} \mathrm{Rep}(\pi_1 M)_{hU} \xrightarrow{\oplus 1} \cdots\right)$$

where $\oplus 1$ denotes the map induced by block sum with the identity matrix $1 \in U(1)$ (note that this induces maps on both the representation space and the universal bundles, hence on homotopy orbit spaces).

We will abbreviate the homotopy colimit in Proposition 4.1.1 by writing

$$\underset{\overrightarrow{\oplus 1}}{\mathrm{hocolim}}(\mathrm{Rep}(\pi_1 M)_{hU}).$$

This result can be proven in a number of ways, including of course by applying Corollary 3.0.16. The starting point for any proof is the McDuff-Segal Group Completion Theorem [33]. Recall from Chapter 3 that so long as π_0 of the right-hand side is a group, or equivalently so long as $\mathrm{Rep}(\pi_1 M)$ is stably group-like with respect to the trivial representation $1 \in \mathrm{Hom}(\pi_1 M, U(1))$ (Proposition 3.0.8), then this theorem provides a zig-zag of maps between the spaces in Proposition 4.1.1, each of which induces an isomorphism on homology with any local (abelian) coefficients. Hence, as in the previous section, the main points are to understand the connected components of the representation spaces, and to show that the fundamental groups on the right (in each component) are abelian.

As we saw in the previous section, this abelianness holds generally in deformation K-theory (Corollary 3.0.16) but for surface groups it can also be seen from Yang-Mills theory. Specifically, the proof of Theorem 4.4.1 shows that the homotopy colimit in Proposition 4.1.1 is weakly equivalent to a space whose fundamental group (in each component) is clearly abelian.

In this thesis, we present two rather different approaches to the problem of showing that $\mathrm{Rep}(\pi_1 M)$ is stably group-like with respect to $1 \in \mathrm{Hom}(\pi_1 M, U(1))$. For the group $\mathbb{Z} = \pi_1 S^1$ this condition is trivially satisfied, since the representation spaces $U(n)$ are connected. In Corollaries 4.3.8 and 4.3.9 we use Yang-Mills theory to show that $\mathrm{Rep}(\pi_1 M)$ is stably group-like for any compact, aspherical surface M. In the orientable case, this amounts to showing that the representation spaces are all connected. This argument is rather close to Ho and Liu's proof of connectivity for the moduli space of flat connections [23, Theorem 20]. For most surfaces, other work of Ho and Liu [25] gives an alternative method, depending on the theory of quasi-Hamiltonian moment maps [4]. A version of their argument, adapted to the present situation, appears in Chapter 6 (Theorem 6.1.9). This method covers all (compact, aspherical) orientable surfaces, but fails two non-orientable ones: the connected sum of either 2 or 4 copies of $\mathbb{R}P^2$. In the former case, we provide an alternative, elementary argument in Propositition 6.1.11. For the latter case, I do not know of a proof that avoids Yang-Mills theory.

Remark 4.1.2 *The monoid* $\mathrm{Rep}(G)_{hU}$ *underlying deformation K-theory is formed using the simplicial model for $EU(n)$ (see Chapter 2), and hence one first obtains a version of Proposition 4.1.1 in which the homotopy orbit spaces*

$$\mathrm{Hom}(\pi_1 M, U(n))_{hU(n)} = EU(n) \times_{U(n)} \mathrm{Hom}(\pi_1 M, U(n))$$

are formed using this simplicial model. In this chapter, we will need to work with classifying spaces of gauge groups, where the simplicial model may not give an honest principal bundle. Hence it is more convenient to use Milnor's model for universal bundles [34], which is functorial and applies to all topological groups. There is a natural zig-zag of weak equivalences connecting these two versions of the classifying space, and this gives a zig-zag connecting the simplicial version of Proposition 4.1.1 *to the Milnor version.*

Generally, if $E_1G \to B_1G$ *and* $E_2G \to B_2G$ *are universal principal* G*-bundles, we set* E_mG *and* B_mG *to be the "mixed models"* $E_mG = E_1G \times E_2G$ *and* $B_mG = E_mG/G$ *(where* G *acts diagonally on* E_mG*). One has a diagram of fibration sequences*

$$\begin{tikzcd}
G \arrow[d] & G \arrow[l, "="'] \arrow[r, "="] \arrow[d] & G \arrow[d] \\
E_1G \arrow[d] & E_mG \arrow[l] \arrow[r] \arrow[d] & E_2G \arrow[d] \\
B_1G & B_mG \arrow[l, "q_1"'] \arrow[r, "q_2"] & B_2
\end{tikzcd}$$

and contractibility of the total spaces implies that the maps q_1 *and* q_2 *are weak equivalences. Now, for any* G*-space* X*, the diagram*

$$\begin{tikzcd}
G \arrow[d] & G \arrow[l, "="'] \arrow[r, "="] \arrow[d] & G \arrow[d] \\
E_1G \times_G X \arrow[d] & E_mG \times_G X \arrow[l] \arrow[r] \arrow[d] & E_2G \times_G X \arrow[d] \\
B_1G & B_mG \arrow[l, "q_1"'] \arrow[r, "q_2"] & B_2
\end{tikzcd}$$

shows that the two versions of the homotopy orbit space are naturally weakly equivalent.

4.2 Representations, flat connections, and semi-stable bundles

Let M denote an n-dimensional, compact, connected manifold, with a fixed basepoint $m_0 \in M$. Let G be a Lie group, and $P \xrightarrow{\pi} M$ be a smooth principal G-bundle, with a fixed basepoint $p_0 \in \pi^{-1}(m_0) \subset P$. Our principal bundles will always have a *right* action of the structure group G.

In this section we explain how to pass from G-representation spaces of $\pi_1(M)$ to spaces of flat connections on principal G-bundles over M, which form critical sets for the Yang-Mills functional. We then explain, in the case when M is a Riemann surface, how Morse theory for the Yang-Mills functional allows one to pass from these critical sets to their stable manifolds. When $G = U(n)$, these stable manifolds consist of semi-stable holomorphic structures on the associated vector bundles.

Before stating the result relating representations to flat connections, we need to introduce the relevant Sobolev spaces of connections and gauge transformations. Our notation and discussion follow [6, Section 14], and another excellent reference is the appendix to [48].

Definition 4.2.1 *Let $k \geqslant 1$ be an integer, and let $1 \leq p < \infty$. We denote the space of all connections on the bundle P of Sobolev class L^p_k by $\mathcal{A}^{k,p}(P)$. This is an affine space, modeled on the Banach space of L^p_k sections of the vector bundle $T^*M \otimes \operatorname{ad} P$ (here $\operatorname{ad} P = P \times_G \mathfrak{g}$, and $\mathfrak{g}$ is the Lie algebra of G equipped with the adjoint action). Hence $\mathcal{A}^{k,p}(P)$ acquires a canonical topology, making it homeomorphic to the Banach space on which it is modeled.*

Flat L^p_k connections are defined to be those with zero curvature. The subspace of flat connections on P is denoted by $\mathcal{A}^{k,p}_{\text{flat}}(P)$.

We let $\mathcal{G}^{k+1,p}(P)$ denote the gauge group of all unitary automorphisms (i.e. gauge transformations) of P of class L^p_{k+1}, and (when $(k+1)p > n$) we let $\mathcal{G}^{k+1,p}_0(P)$ denote the subgroup of based automorphisms (those which are the identity on the fiber over $m_0 \in M$). These gauge groups are Banach Lie groups, and act smoothly on $\mathcal{A}^{k,p}(P)$. We will always use the left action, meaning that we let gauge transformations act on connections by pushforward. We denote the group of all continuous gauge transformations by $\mathcal{G}(P)$. Note that so long as $(k+1)p > n$, the Sobolev Embedding Theorem gives a continuous inclusion $\mathcal{G}^{k+1,p}(P) \hookrightarrow \mathcal{G}(P)$, and hence in this range the based gauge group is well-defined.

We denote the smooth versions of these objects by $(-)^\infty$, and when the bundle P is trivial, we will use the notation $\mathcal{A}^{k,p}(n) = \mathcal{A}^{k,p}(M \times U(n))$, and so on.

Remark 4.2.2 *We use the notation L^p_k to denote functions with k weak (distributional) derivatives, each in the Sobolev space L^p.*

We will record the necessary assumptions on k and p as they arise. The reader interested only in the applications to deformation K-theory may safely ignore these *issues, noting only that all the results of this section hold in the Hilbert space L^2_k for large enough k. When $n = 2$, our main case of interest, we just need $k \geqslant 2$.*

The following lemma is well-known.

Lemma 4.2.3 *Assume $(k+1)p > n$. Then the inclusion $\mathcal{G}^{k+1,p}(P) \hookrightarrow \mathcal{G}(P)$ is a weak equivalence.*

Proof. A gauge transformation is simply a section of the adjoint bundle $P \times_G \mathrm{Ad}(G)$, where $\mathrm{Ad}(G)$ denotes the group G equipped with the adjoint action of G on itself (see [6, Section 2]). Hence this result follows from general approximation results for sections of smooth fiber bundles. □

Note the continuous inclusion $\mathcal{G}^{k+1,p}(P) \hookrightarrow \mathcal{G}(P)$ implies that there is a well-defined, continuous homomorphism $r : \mathcal{G}^{k+1,p}(P) \to G$ given by restricting a gauge transformation to the fiber over the basepoint $m_0 \in M$. To be precise, $r(\phi)$ is defined by $p_0 \cdot r(\phi) = \phi(p_0)$, and hence depends on our choice of basepoint $p_0 \in P$.

Lemma 4.2.4 *If G is connected, then the restriction map $r : \mathcal{G}^{k+1,p}(P) \longrightarrow G$ is surjective. If we assume further that $(k+1)p > n$, then r induces a homeomorphism $\mathcal{G}^{k+1,p}(P)/\mathcal{G}_0^{k+1,p}(P) \cong G$. The same statements hold for the smooth gauge group.*

Proof. Thinking of gauge transformations as sections of the adjoint bundle, we may deform the identity map $P \to P$ over a neighborhood of m_0 so that it takes any desired value at p_0 (here we use, of course, connectivity of G). This proves surjectivity.

By a similar argument, we may construct continuous local sections $s : U \to \mathcal{G}^\infty(P)$ of the map r, where $U \subset G$ is any chart. If $\pi : \mathcal{G}^\infty(P) \to \mathcal{G}(P)^\infty/\mathcal{G}_0^\infty(P)$ is the quotient map, then the maps $\pi \circ s$ are inverse to $\overline{r}$ on U. Hence $\overline{r}^{-1}$ is continuous. The same argument applies to $\mathcal{G}^{k+1,p}(P)$, although we must require $(k+1)p > n$ so that r is well-defined and continuous. □

I do not know whether Lemma 4.2.4 holds for non-connected groups; certainly the proof shows that the image of the restriction map is always a union of components of G.

Flat connections are related to representations of $\pi_1 M$ via the holonomy map. Our next goal is to analyze this map carefully in the current context of Sobolev connections. The holonomy of a smooth connection is defined via parallel transport: given a smooth loop γ based at $m_0 \in M$, there is a unique A-horizontal lift $\widetilde{\gamma}$ of γ with $\widetilde{\gamma}(0) = p_0$, and the holonomy representation $\mathcal{H}(A) = \rho_A$ is then defined by the equation $\widetilde{\gamma}(1) \cdot \rho_A([\gamma]) = p_0$. Since flat connections are locally trivial, a standard compactness argument shows that this definition depends only on the homotopy class of γ. It is important to note here that the holonomy map depends on the chosen the basepoint $p_0 \in P$. For further details on holonomy, we refer the reader to Appendix A.

Lemma 4.2.5 *The holonomy map $\mathcal{A}^{k,p}_{\text{flat}}(P) \to \text{Hom}(\pi_1 M, G)$ is continuous if $k \geqslant 2$ and $(k-1)p > n$.*

Proof. By the Sobolev Embedding Theorem, the assumptions on k and p guarantee a continuous embedding $L^p_k(M) \hookrightarrow C^1(M)$. Hence if $A_i \in \mathcal{A}^{k,p}_{\text{flat}}(P)$ is a sequence of connections converging (in $\mathcal{A}^{k,p}_{\text{flat}}(P)$) to A, then $A_i \to A$ in C^1 as well. We must show that for any such sequence, the holonomies of the A_i converge to the holonomy of A.

Let $\gamma_1, \ldots, \gamma_m$ be smooth curves in M which generate $\pi_1 M$. Since the topology on $\text{Hom}(\pi_1 M, G)$ is the restriction of the product topology on G^m, it suffices to check that for each i the holonomies around γ_i converge. Let γ be one of the γ_i. Then each connection A_i pulls back to define a vector field V_i on $\gamma^* P$, and the holonomy of A_i along γ is defined (continuously) in terms of the integral curves of this vector field. Hence it suffices to show that the integral curves of the vector fields V_i converge to those of the vector field V associated to A. We may assume that the C^1 norms $t_i = ||V_i - V||_{C^1}$ are decreasing and less than 1. By interpolating linearly between the V_i, we obtain a vector field on $\gamma^* P \times I$ which at time t_i is just V_i, and at time 0 is V. This is clearly a Lipschitz vector field and hence the integral curves vary continuously in the initial point [28], completing the proof. □

Remark 4.2.6 *With a bit more care, one can prove Lemma 4.2.5 under the weaker assumptions $k \geqslant 1$ and $kp > n$. The basic point is that these assumptions give an embedding $L^p_k(M) \hookrightarrow C^0(M)$, and by compactness $C^0(M) \hookrightarrow L^1(M)$ (and similarly* *after restricting to a smooth curve in M). Working in local coordinates, one can deduce continuity of the holonomy map from the fact that limits commute with integrals in $L^1([0,1])$.*

Lemma 4.2.7 *Assume $p > n/2$ (and if $n = 2$, assume $p \geqslant 4/3$). If G is connected, then each $\mathcal{G}_0^{k+1,p}(P)$-orbit in $\mathcal{A}_{\text{flat}}^{k,p}(P)$ contains a unique $\mathcal{G}_0^{\infty}(P)$-orbit of smooth connections.*

Proof. The assumptions on k and p guarantee that each $\mathcal{G}^{k+1,p}(n)$ orbit in $\mathcal{A}_{\text{flat}}^{k,p}(n)$ contains a smooth connection: this is a special case of a result in the theory of Uhlenbeck Compactness, which seems to have first been explicitly proven by Wehrheim [48, Theorem 9.4]. Now, say $\phi \cdot A$ is smooth for some $\phi \in \mathcal{G}^{k+1,p}(P)$. By Lemma 4.2.4, there exists a smooth gauge transformation ψ such that $r(\psi) = r(\phi)^{-1}$.

Now $\psi \circ \phi$ is clearly based, and since ψ is smooth we know that $(\psi \circ \phi) \cdot A$ is still smooth. This proves existence. For uniqueness, say $\phi \cdot A$ and $\psi \cdot A$ are both smooth, where $\phi, \psi \in \mathcal{G}_0^{k+1,p}(P)$. Then $\phi^{-1}\psi$ is smooth by [6, Lemma 14.9], so these connections lie in the same $\mathcal{G}_0^{\infty}$-orbit. $\square$

We can now prove the result which connects representation theory with Yang-Mills theory.

Proposition 4.2.8 *Assume $p > n/2$ (and if $n = 2$, assume $p > 4/3$). Assume also that $kp > n$. Then for any n-manifold M and any compact, connected Lie group G, the holonomy map induces a G-equivariant homeomorphism*

$$\coprod_{[P^n]} \mathcal{A}_{\text{flat}}^{k,p}(P^n)/\mathcal{G}_0^{k+1,p}(P^n) \xrightarrow{\overline{\mathcal{H}}} \operatorname{Hom}(\pi_1(M), G),$$

where the disjoint union is taken over some set of representatives for the (unbased) isomorphism classes of principal G-bundles over M. (Note that to define $\overline{\mathcal{H}}$ we choose, arbitrarily, base points in each representative bundle P^n.)

The G-action on the left is induced by the actions of $\mathcal{G}^{k+1,p}(P^n)$ together with the homeomorphism $\mathcal{G}^{k+1,p}(P^n)/\mathcal{G}_0^{k+1,p}(P^n) \cong G$, which again depends on chosen basepoints in the bundles P^n.

Proof. The assumptions on k and p allow us to employ all previous results in this section (note Remark 4.2.6). It is well-known that the holonomy map

$$H : \coprod_{[P^n]} \mathcal{A}^{\infty}_{\text{flat}}(P^n) \longrightarrow \text{Hom}(\pi_1(M), G)$$

is invariant under the action of the based gauge group and induces an equivariant bijection

$$\bar{H} : \coprod_{[P^n]} \mathcal{A}^{\infty}_{\text{flat}}(P^n)/\mathcal{G}^{\infty}_0(P^n) \longrightarrow \text{Hom}(\pi_1(M), G).$$

(See Appendix A for a detailed proof.) By Lemma 4.2.7, the left hand side is unchanged (set-theoretically) if we replace $\mathcal{A}^{\infty}_{\text{flat}}$ and $\mathcal{G}^{\infty}_0$ by $\mathcal{A}^{k,p}_{\text{flat}}$ and $\mathcal{G}^{k+1,p}_0$, and hence Lemma 4.2.5 tells us that we have a continuous equivariant bijection

$$\bar{\mathcal{H}} : \coprod_{[P^n]} \mathcal{A}^{k,p}_{\text{flat}}(P^n)/\mathcal{G}^{k+1,p}_0(P^n) \longrightarrow \text{Hom}(\pi_1(M), G).$$

(We note that the proof of Proposition A.0.15, although written in terms of smooth connections, in fact goes through equally well in the present setting and proves that holonomy is invariant under the based gauge group $\mathcal{G}^{k+1,p}_0(P^n)$.)

We will show that for each P, $\mathcal{A}^{k,p}_{\text{flat}}(P)/\mathcal{G}^{k+1,p}_0(P)$ is sequentially compact. Since, by Proposition A.0.30, only finitely many isomorphism types of principal G-bundle admit flat connections, this will imply that

$$\coprod_{[P^n]} \mathcal{A}^{k,p}_{\text{flat}}(P^n)/\mathcal{G}^{k+1,p}_0(P^n)$$

is sequentially compact. Since a continuous bijection from a sequentially compact space to a Hausdorff space is a homeomorphism (see Lemma 4.2.9 below) this will complete the proof.

The Strong Uhlenbeck Compactness Theorem [48] (see also [12, Proposition 4.1]) states that the space $\mathcal{A}^{k,p}_{\text{flat}}(P)/\mathcal{G}^{k+1,p}(P)$ is sequentially compact. Now, given a sequence $\{A_i\}$ in $\mathcal{A}^{k,p}_{\text{flat}}(P)$, there exists a sub-sequence $\{A_{i_j}\}$ and a sequence $\phi_j \in \mathcal{G}^{k+1,p}(n)$ such that $\phi_j \cdot A_{i_j}$ converges in $\mathcal{A}^{k,p}$ to a flat connection A. Let $g_j = r(\phi_j)$. Since G is compact, passing to a sub-sequence if necessary we may assume that the g_j converge to an element $g \in G$. The proof of Lemma 4.2.4 shows that we may choose a convergent sequence $\psi_j \in \mathcal{G}^{k+1,p}(P)$ such that $r(\psi_j) = g_j^{-1}$; we let $\psi = \lim \psi_j$, so $r(\psi) = g^{-1}$. Now continuity of the action implies that the sequence $(\psi_j \circ \phi_j) \cdot A_{i_j}$ converges to $\psi \cdot A$. Since $\psi_j \circ \phi_j \in \mathcal{G}_0^{k+1,p}(P)$, this completes the proof. $\square$

For the above proof, we needed the following elementary lemma:

Lemma 4.2.9 *Let $f : X \to Y$ be a continuous bijection. If X is sequentially compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. We must show that f is a closed map. First, note that any closed subset of a sequentially compact space is sequentially compact. Now, if C is closed in X we must show that $f(C)$ is closed in Y. But the continuous image of a sequentially compact space is sequentially compact, and sequentially compact subsets of Hausdorff spaces are closed. $\square$

Remark 4.2.10 *It is worth noting that point-set considerations alone show that sequential compactness of the quotient space $\mathcal{A}_{\text{flat}}^{k,p}(P)/\mathcal{G}_0^{k+1,p}(P)$ suffices to prove its compactness: specifically, $\mathcal{A}_{\text{flat}}^{k,p}(P)$ is second countable, since it is a subspace of a separable Banach space. Since the quotient map of a group action is always open, we may conclude that $\mathcal{A}_{\text{flat}}^{k,p}(P)/\mathcal{G}_0^{k+1,p}(P)$ is second countable as well. Now, any space which is first countable and sequentially compact is countably compact [50, 7.1.3], and any second countable space is Lindelöf [50, 5.3.2]. Finally, any countably compact Lindelöf space is compact.*

More interesting is the fact that Proposition 4.2.8 implies that the based gauge orbits in $\mathcal{A}_{\text{flat}}^{k,p}(P)$ are closed (because the quotient is homeomorphic to the Hausdorff space $\mathrm{Hom}(\pi_1 M, G)$*). Since the homeomorphism is G-equivariant and G is compact, one also concludes that the full gauge orbits are closed.*

The following is a slightly more direct version of [6, 13.1].

Lemma 4.2.11 *Let $\mathcal{G}$ be a topological group, acting on a space X. Assume that $\mathcal{N} \lhd \mathcal{G}$ acts freely on X, with $X \to X/N$ a principal N-bundle. Then the natural map*

$$E\mathcal{G} \times_{\mathcal{G}} X \longrightarrow E(\mathcal{G}/\mathcal{N}) \times_{\mathcal{G}/\mathcal{N}} X/\mathcal{N}$$

is a weak equivalence.

Proof. The group $\mathcal{G}$ acts freely on $E(\mathcal{G}/\mathcal{N})\times X$ via $(e,x)\cdot g = (e\cdot\pi(g), g^{-1}\cdot x)$, where $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{N}$ is the quotient map. The quotient map factors as the composite

$$E(\mathcal{G}/\mathcal{N}) \times X \longrightarrow E(\mathcal{G}/\mathcal{N}) \times X/\mathcal{N} \longrightarrow E(\mathcal{G}/\mathcal{N}) \times_{\mathcal{G}/\mathcal{N}} X/\mathcal{N}.$$

Each of these maps is a fibration (in fact, a fiber bundle) so the composite is (at least) a fibration. The lemma now follows from the diagram of fibrations

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d, "\|"'] & E\mathcal{G} \times X \arrow[r] \arrow[d, "\wr"] & E\mathcal{G} \times_{\mathcal{G}} X \arrow[d] \\
\mathcal{G} \arrow[r] & E(\mathcal{G}/\mathcal{N}) \times X \arrow[r] & E(\mathcal{G}/\mathcal{N}) \times_{\mathcal{G}/\mathcal{N}} X/\mathcal{N}.
\end{tikzcd}$$

□

Corollary 4.2.12 *Assume $p > n/2$ (and if $n = 2$, assume $p > 4/3$). Assume also that $kp > n$. If the structure group G is compact and connected, then the natural projection map*

$$\coprod_{[P^n]} E\mathcal{G}^{k+1}(P^n) \times_{\mathcal{G}^{k+1}(P^n)} \mathcal{A}^1_{\text{flat}}(P^n) \xrightarrow{\cong} EG \times_G \operatorname{Hom}(\pi_1(M), G)$$

is a weak equivalence.

Proof. The based gauge groups $\mathcal{G}_0^{k+1,p}(P^n)$ acts freely on $\mathcal{A}^{k,p}(P^n)$, and the projection maps are locally trivial principal $\mathcal{G}_0^{k+1,p}(P^n)$-bundles [35]. Hence the same is true when we restrict to the invariant subspaces of flat connections. Since $\mathcal{G}^{k+1,p}(P^n)/\mathcal{G}_0^{k+1,p}(P^n) \cong G$ and

$$\coprod_{[P^n]} \mathcal{A}^{k,p}_{\text{flat}}(P^n)/\mathcal{G}_0^{k+1,p}(P^n) \xrightarrow{\cong} \operatorname{Hom}(\pi_1 M, G)$$

(Lemma 4.2.4 and Proposition 4.2.8), the result follows from Lemma 4.2.11. □

We now focus on the case where M is a compact Riemann surface and $G = U(n)$. It is best here to work in the Hilbert space of L^2_k connections, and we must assume

$k \geqslant 2$ so that the results of this section apply. We will now suppress $p = 2$ from the notation, writing simply $\mathcal{A}^k$, $\mathcal{G}^k$, and so on. Additionally, we will see in the proof of Corollary 4.3.8 that if P is a principal $U(n)$-bundle over a Riemann surface with $\mathcal{A}_{\text{flat}}(P) \neq \emptyset$, then P is trivial. Hence we may restrict our attention to the case $P = M \times U(n)$, and we use the notation $\mathcal{A}^k(n) = \mathcal{A}^k(M \times U(n))$, and similarly for the gauge groups.

Our next goal is to pass from the spaces $\mathcal{A}^k_{\text{flat}}(n)$ to the larger space $\mathcal{C}^k_{ss}(n)$ consisting of semi-stable holomorphic structures on the associated vector bundle $M \times \mathbb{C}^n$.

The set $\mathcal{C}(E)$ of holomorphic structures on a complex vector bundle E may be viewed as an affine space, modeled on the vector space $\Omega^{0,1}(M; \operatorname{End} E)$ of endomorphism-valued $(0,1)$-forms (see [6, Sections 5, 7]). Since this is the space of (smooth) sections of a vector bundle on M, we may define Sobolev spaces $\mathcal{C}^k(E) = \mathcal{C}^{k,2}(E)$ of holomorphic structures simply by taking L^2_k-sections of this bundle. If we fix a Hermitian metric on E, then to each holomorphic structure there corresponds a unique compatible (metric) connection [17, p. 73]. When M is a Riemann surface, this induces an isomorphism of affine spaces, which extends to an isomorphism $\mathcal{A}^k(P) \cong \mathcal{C}^k(P \times_{U(n)} \mathbb{C}^n)$. For further details, see the references cited above or [12, Section 2].

Definition 4.2.13 *A holomorphic bundle E over M is semi-stable if for every proper holomorphic sub-bundle $E' \subset E$, one has*

$$\frac{\deg(E')}{\operatorname{rk}(E')} \leqslant \frac{\deg(E)}{\operatorname{rk}(E)}.$$

(Replacing the $\leqslant$ by $<$ in this definition, one has the definition of a stable *bundle.) Here* $\deg(E)$ *refers to the first Chern number of the bundle, i.e. the integer obtained by evaluating the first Chern class $c_1(E)$ on the fundamental class of the (oriented) Riemann surface M.*

As we will explain, the space of semi-stable bundles (or rather its Sobolev analogue) plays the role of the stable manifold for the space of flat connections, which is a critical set of the Yang-Mills functional L.

For any smooth principal $U(n)$-bundle $P \to M$, the Yang-Mills functional $L : \mathcal{A}^k(P) \to \mathbb{R}$ is defined by the formula

$$L(A) = \int_M ||F(A)||^2 d\text{vol}$$

where $F(A)$ denotes the curvature form of the connection A and the volume of M is normalized to be 1. Here $|| \cdot ||$ refers to a natural Riemannian metric on the bundle $T^*M \otimes T^*M \otimes ad(P)$; note that $F(A)$ is a section of this bundle so we may apply the Riemannian metric pointwise to $F(A)$. (For a construction of this metric, see [15].)

Råde has shown [40] that the gradient flow of the Yang-Mills functional produces a deformation retraction of the Morse strata (defined via the flow of the Yang-Mills functional) onto their critical sets. Daskalopoulos has shown [12, Theorem 6.2] that the Morse stratification of $\mathcal{A}^k(P)$ coincides with the Harder-Narasimhan stratification. The latter stratification, defined in the next section, exists initially on the space of smooth holomorphic structures, but can of course be transported to the isomorphic space of smooth connections. It extends to the space of L^2_k holomorphic structures (or connections) because each such holomorphic structure is gauge-equivalent to a smooth connection, and the smooth gauge-equivalence class of this smooth connection is well-defined. Here we need to use the complex gauge group; see [6, Section 14].

As discussed in Section 4.3, the space $\mathcal{C}^k_{ss}(n)$ of semi-stable holomorphic structures (on the trivial bundle $M \times \mathbb{C}^n$) is one of the Harder-Narasimhan strata, and its critical set is precisely the space of flat connections. To see that every flat connection corresponds to a semi-stable bundle, one uses the Narasimhan-Seshardri Theorem [6, (8.1)], which says that irreducible representations induce *stable* bundles. By Proposition 4.2.8, every flat connection comes from some unitary representation, which is a sum of irreducible representations, and hence the holomorphic bundle associated to any representation, i.e. any flat connection, is a sum of stable bundles. But it is an elementary fact that an extension of stable bundles of the same degree is always semi-stable.

In summary, we have:

Theorem 4.2.14 (Daskalopoulos, Rade) *Let M be a compact Riemann surface. Then there is a continuous deformation retraction from the space $\mathcal{C}^k_{ss}(n)$ of all semi-stable L^2_k holomorphic structures on $M \times \mathbb{C}^n$ to the subspace $\mathcal{A}^k_{\text{flat}}(n)$ of flat (unitary) connections.*

We will discuss the analogue of this result in the non-orientable case in the proof of Proposition 4.3.7.

4.3 The Harder-Narasimhan stratification

In the previous section, we explained how to pass from spaces of representations to spaces of semi-stable holomorphic structures. The next step will be to pass from semi-stable structures to the affine space of all holomorphic structures (or, equivalently, all connections). Although for any finite n there is a substantial difference between these spaces, this difference will disappear when we pass to the limit (by adding trivial holomorphic lines).

We fix a Riemann surface M of genus g and suppress g from the notation when possible.

We now introduce the Harder-Narasimhan stratification [6, Section 7] on the space $\mathcal{C}^k(n)$ of holomorphic structures on a trivial rank n vector bundle over M. This stratification is induced from a stratification on the subspace $\mathcal{C}(n)$ of smooth structures, via the fact that each $a \in \mathcal{C}^k(n)$ is gauge-equivalent to a unique (smooth) orbit of smooth structures (this result is proven in [6, Section 14]; here the gauge transformations may lie in the *complex* gauge group of vector bundle automorphisms).

Given a (smooth) holomorphic structure $\mathcal{E}$ on the bundle $M \times \mathbb{C}^n$, there is a unique filtration (the Harder-Narasimhan filtration [19])

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \mathcal{E}_r = \mathcal{E}$$

of $\mathcal{E}$ by holomorphic sub-bundles with the property that each quotient $\mathcal{D}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ is semi-stable ($i = 1, \ldots, r$) and $\mu(\mathcal{D}_1) > \mu(\mathcal{D}_2) > \cdots > \mu(\mathcal{D}_r)$, where $\mu(\mathcal{D}_i) =$

$\frac{\deg(\mathcal{D}_i)}{\mathrm{rank}(\mathcal{D}_i)}$, and $\deg(\mathcal{D}_i)$ is the first Chern number of the vector bundle $\mathcal{D}_i$. Letting $n_i = \mathrm{rank}(D_i)$ and $k_i = \deg(D_i)$, we call the sequence

$$\mu = ((n_1, k_1), \ldots, (n_r, k_r))$$

the *type* of $\mathcal{E}$. Let $\mathcal{C}^k_\mu = \mathcal{C}^k_\mu(n) \subset \mathcal{C}^k(n)$ denote the subspace of all holomorphic structures gauge-equivalent to a smooth structure of type μ. Note that the semi-stable stratum corresponds to $\mu = ((n, 0))$, and that since degrees add in exact sequences we have $\sum_i k_i = 0$.

The following definition will be useful.

Definition 4.3.1 *Consider a sequence of pairs of integers* $((n_1, k_1), \ldots, (n_r, k_r))$. *We call such a sequence* admissible *of total rank* n *(and total Chern class* 0*) if* $n_i > 0$ *for each* i, $\sum n_i = n$, $\sum_i k_i = 0$, *and* $\frac{k_1}{n_1} > \cdots > \frac{k_r}{n_r}$. *Hence admissible sequences of total rank* n *and total Chern class 0 are precisely those describing Harder-Narasimhan strata in* $\mathcal{C}(n)$.

We denote the collection of all admissible sequences of total rank n *and total Chern class* 0 *by* $\mathcal{I}(n)$.

With this notation, we now have the following result from [6, Section 7] (see also [12, Theorem B]).

Theorem 4.3.2 *Let* $\mu = ((n_1, k_1), \ldots, (n_r, k_r)) \in \mathcal{I}(n)$. *Then the stratum* $\mathcal{C}^k_\mu$ *is a locally closed submanifold of* $\mathcal{C}^k(n)$ *with complex codimension given by*

$$c(\mu) = \left(\sum_{i>j} n_i k_j - n_j k_i \right) + (g-1) \left(\sum_{i>j} n_i n_j \right).$$

We now introduce useful way of thinking about the Harder-Narasimhan strata (due to Shatz [43], see also [6, Section 7]). Given an admissible sequence μ, we can construct a convex path $P(\mu)$ in the plane starting at $(0, 0)$ and ending at $(n, 0)$ by connecting the points $(\sum_{j=1}^i n_j, \sum_{j=1}^i k_j)$ with straight lines ($i = 1, 2, \cdots n$). Convexity of the path corresponds precisely to the condition that the slopes of

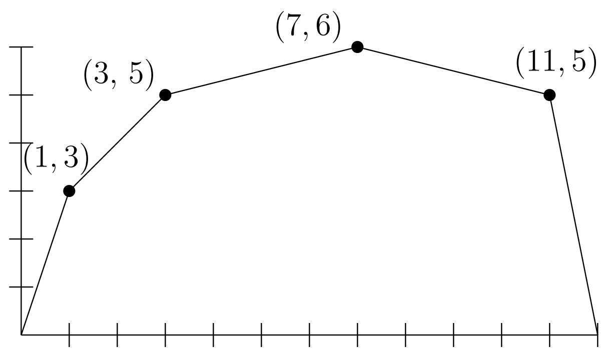

Figure 4.1: The convex path $P((1,3),(2,2),(4,1),(4,-1),(1,-5))$.

these lines decrease, i.e. that

$$\frac{k_1}{n_1} > \frac{k_2}{n_2} > \cdots > \frac{k_r}{n_r}.$$

See Figure 4.1 for an example.

We now compute the minimum codimension of a non semi-stable stratum. In particular, this computation shows that this minimum tends to infinity with n, so long as the genus g is positive.

Lemma 4.3.3 *The minimum (real) codimension of a non semi-stable stratum in $\mathcal{C}^k(n)$ $(n > 1)$ is precisely* $2n + 2(n-1)(g-1) = 2g(n-1) + 2$.

Proof. Let $\mu = ((n_1, k_1), \ldots, (n_r, k_r)) \in \mathcal{I}(n)$ be any admissible sequence with $r > 1$. Then from Theorem 4.3.2, we see that it will suffice to show that

$$\sum_{i>j} n_i k_j - n_j k_i \geqslant n \tag{4.1}$$

and

$$\sum_{i>j} n_i n_j \geqslant n - 1. \tag{4.2}$$

To prove (4.1), we begin by noting that since $\sum k_i = 0$ and the slopes $\frac{k_i}{n_i}$ are strictly decreasing, we must have $k_1 > 0$ and $k_n < 0$ (in terms of convex paths, this simply says that a convex path from $(0,0)$ to $(0,n)$ must have positive initial slope and negative final slope). Moreover, there is some $l_0 \in \mathbb{R}$ such that $k_l \geqslant 1$ for $l < l_0$

and $k_l \leqslant -1$ for $l > l_0$ (this number just indicates when the path $P(\mu)$ switches from increasing to decreasing). We allow l_0 to be an integer if and only if $k_l = 0$ for some l; then this integer l is unique, and we set $l_0 := l$. Since $r \geqslant 2$, we know that $1 < l_0 < r$.

Now, if $i > l_0 > j$ we have $k_j \geqslant 1$ and $k_i \leqslant -1$, so,

$$n_i k_j - n_j k_i \geqslant n_i + n_j.$$

If $i > l_0$ and $j = l_0$, we have $k_j = 0$ and $k_i \leqslant -1$, so

$$n_i k_{l_0} - n_{l_0} k_i \geqslant 0 + n_{l_0} = n_{l_0}.$$

Finally, if $i = l_0$ and $j < l_0$, then $k_i = 0$ and $k_j \geqslant 1$ so we have

$$n_{l_0} k_j - n_j k_{l_0} \geqslant n_{l_0} - 0 = n_{l_0}.$$

Now, since $n_i k_j - n_j k_i = n_i n_j (k_j/n_j - k_i/n_i)$ and the slopes k_l/n_l are strictly decreasing, we know that each term in the sum $\sum_{i>j} n_i k_j - n_j k_i$ is positive. Dropping terms and applying the above bounds gives

$$\begin{aligned}\sum_{i>j} n_i k_j - n_j k_i &\geqslant \sum_{i>l_0>j} (n_i k_j - n_j k_i) + \sum_{l_0>j} (n_{l_0} k_j - n_j k_{l_0}) + \sum_{i>l_0} (n_i k_{l_0} - n_{l_0} k_i) \\ &\geqslant \sum_{i>l_0>j} (n_i + n_j) + \sum_{l_0>j} n_{l_0} + \sum_{i>l_0} n_{l_0}.\end{aligned}$$

(In the second and third expressions, the latter sums are taken to be empty if l_0 is not an integer.) Since $\sum n_i = n$, to check that the above expression is at least n it suffices to check that each n_i appears in the final sum. But since $1 < l_0 < r$, each n_l with $l \neq l_0$ appears in the first term, and if $l_0 \in \mathbb{N}$ then n_{l_0} appears in both of the latter terms. This completes the proof of (4.1).

To prove (4.2), we fix $r \in \mathbb{N}$ ($r \geqslant 2$) and consider partitions $\overrightarrow{\mathbf{p}} = (p_1, \ldots, p_r)$ of n. We will minimize the function $\phi_r(\overrightarrow{\mathbf{p}}) = \sum_{i>j} p_i p_j$, over all length r partitions of n. (It is useful to note that the sum defining ϕ_r is taken over all 2-element subsets of $\{1, \ldots, r\}$; the condition $i > j$ is simply convenient notation.)

Consider a partition $\vec{\mathbf{p}}= (p_1, \ldots, p_r)$ with $p_m \geqslant p_l > 1$ $(l \neq m)$, and define another partition $\vec{\mathbf{p}'}$ by setting

$$p'_i = \begin{cases} p_i, & i \neq l,\ m, \\ p_l - 1, & i = l, \\ p_m + 1 & i = m. \end{cases}$$

We claim that

$$\phi_r(\vec{\mathbf{p}}) > \phi_r(\vec{\mathbf{p}'}).$$

Indeed, the right hand side is

$$(p_l - 1)(p_m + 1) + \sum_{i,j\neq l,m;\ i>j} p_i p_j + \sum_{j \neq l,m} (p_l - 1)p_j + \sum_{i\neq l,m} p_i(p_m + 1)$$

$$= p_l - p_m - 1 + \sum_{i>j} p_i p_j - \sum_{j\neq l,m} p_j + \sum_{i \neq l,m} p_i$$

$$= p_l - p_m - 1 + \sum_{i>j} p_i p_j,$$

and since $p_m \geqslant p_l$, we have $p_l - p_m - 1 < 0$.

Now, if we start with any partition $\vec{\mathbf{p}}$ such $p_i > 1$ for more than one index i, the above argument shows that $\vec{\mathbf{p}}$ cannot minimize ϕ_r. Thus ϕ_r is minimized by the partition $\vec{\mathbf{p}}_0= (1, \ldots, 1, n - r - 1)$, and $\phi_r(\vec{\mathbf{p}}_0) = \binom{r-1}{2} + (r - 1)(n - r - 1)$. If we let $f(r) = \binom{r-1}{2} + (r - 1)(n - r - 1)$, then we see that this is an increasing function on the interval $(0, n)$ and hence for partitions of length at least 2, the formula $\sum_{i>j} p_i p_j$ is minimized by the length 2 partition $(1, n - 1)$. In this case, of course, we obtain the desired lower bound of $n - 1$. This completes the proof of (4.2).

To complete the proof of the lemma we must exhibit, for each $n \geqslant 2$, an admissible sequence μ with complex codimension $n+(n-1)(g-1)$. This is the sequence $((1, 1), (n - 1, -1))$. □

Remark 4.3.4 *It is interesting to note that the results in the next section clearly fail in the case when M has genus 0. In this case, M is the sphere, so $\pi_1 M = 0$ and $\mathrm{Hom}(\pi_1 M, U(n))$ is a point. From the point of view of homotopy theory, the* *problem is that S^2 is not the classifying space of its fundamental group, and so one should not expect a relationship between K-theory of S^2 and representations of $\pi_1 S^2 = 0$. But the only place where our argument breaks down in the genus 0 case is the previous lemma, which tells us that there are strata of complex codimension 1 in the Harder-Narasimhan stratification of $\mathcal{C}^k(S^2 \times \mathbb{C}(n))$, and in particular the minimum codimension does not tend to infinity with the rank. Thus there appears to be a relationship between the codimensions of these strata and the contractibility of the universal cover of M. It is difficult, however, to formulate a general conjecture (for 3-manifolds, rather than surfaces, say). Letting the rank tend to infinity must be replaced by a different sort of limiting process, governed by the structure of the monoid $\pi_0 Rep(\pi_1 M)$. The reason we let the rank tend to infinity in the surface case is that this monoid is stably group-like with respect to the trivial representation $1 \in \mathrm{Hom}(\pi_1 M, U(1))$ (or more specifically, each representation space $\mathrm{Hom}(\pi_1 M, U(n))$ is connected), and block sum with this representation corresponds to increasing the rank. There is no reason to expect $\pi_0 Rep(\pi_1 M)$ to be so simple when M is a 3-manifold.*

The main result of this section will be an application of the following infinite-dimensional transversality theorem, due to Smale [2, Theorem 19.1] (see also [1]). Recall that a residual set in a topological space is a countable intersection of open, dense sets. By the Baire category theorem, any residual subset of a Banach space is dense, and since any Banach manifold is *locally* a Banach space, any residual subset of a Banach manifold is dense as well.

Theorem 4.3.5 (Smale) *Let A, X, and Y be second countable C^r Banach manifolds, with X of finite dimension k. Let $W \subset Y$ be a (locally closed) submanifold of Y, of finite codimension q. Assume that $r > \max(0, k-q)$. Let $\rho : A \to C^r(X, Y)$ be a C^r-representation, that is, a function for which the evaluation map $\mathrm{ev}_\rho : A \times X \to Y$ given by $\mathrm{ev}_\rho(a, x) = \rho(a)x$ is of class C^r.*

For $a \in A$, let $\rho_a : X \to Y$ be the map $\rho_a(x) = \rho(a)x$. Then $\{a \in A | \rho_a \pitchfork W\}$ is residual in A, provided that $\mathrm{ev}_\rho \pitchfork W$.

Corollary 4.3.6 *Let Y be a second countable Banach space, and let $\{W_i\}_{i \in I}$ be a collection of (locally closed) submanifolds of Y with finite codimension (we need* *not assume the W_i are disjoint). Then if $U = Y - \bigcup_{i\in I} W_i$ is non-empty, it has connectivity at least $\mu - 2$, where*

$$\mu = \min\{\text{codim } W_i \ : \ i \in I\};$$

equivalently the inclusion $U \hookrightarrow Y$ is $(\mu - 2)$-connected.

Proof. To begin, consider a continuous map $f : S^{k-1} \to U$, with $k - 1 \leqslant \mu - 2$. We must show that f is null-homotopic in U; note that our homotopy need not be based. First we note that f may be smoothed, i.e. we may replace f by a homotopic map which is of class C^{k+1}.

Choose a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ with the property that $\phi(t) = 1$ for $t \geqslant 1/2$ and ϕ vanishes to all orders at 0. Let $D^k \subset \mathbb{R}^{k+1}$ denote the closed unit disk, so $\partial D^k = S^{k-1}$. The formula $H^+(x) = \phi(||x||)f(x/||x||)$ gives a C^{k+1} map $D^k \to Y$ which restricts to f on each shell $\{x \in D^k \mid \ ||x|| = r\}$ with $r \geqslant 1/2$. In particular, H^+ defines a C^{k+1} null-homotopy of f. We may now define another C^{k+1} map $H : S^k \to Y$ by gluing two copies of the map H^+. (The point of this construction is to obtain a "null-homotopy" which is defined on a compact manifold without boundary.)

We now define

$$A = \{F \in C^{k+1}(S^k, Y) \mid \ F(x) = 0 \ \text{ for } \ x \in S^{k-1} \subset S^k\}.$$

Note that A is a Banach space: since S^k is compact, [1, Theorem 5.4] implies that $C^{k+1}(S^k, Y)$ is a Banach space, and A is a closed subspace of $C^{k+1}(S^k, Y)$. (This is the reason for working with C^{k+1} maps rather than smooth ones.)

Next, we define $\rho : A \to C^{k+1}(S^k, Y)$ by setting $\rho(F) = F + H$. The evaluation map $\text{ev}_\rho : A \times S^k \to Y$ is given by $\text{ev}_\rho(F, x) = F(x) + H(x)$. Since both $(F, x) \mapsto F(x)$ and $(F, x) \mapsto x \mapsto H(x)$ are of class C^{k+1}, so is their sum (the fact that the evaluation map $(F, x) \mapsto F(x)$ is of class C^{k+1} follows from [1, Lemma 11.6]).

We are now ready to apply the transversality theorem. Setting $X = S^k$, $W = W_i$ (for some $i \in I$) and with A as above, all the hypotheses of Theorem 4.3.5 are clearly satisfied, except for the final requirement that $\text{ev}_\rho \pitchfork W_i$. But this is easily seen to be the case. In fact, the derivative of ev_ρ surjects onto T_yY for each y in the image of ev_ρ, because given a C^{k+1} map $F : S^k \to Y$ with $F(x) = y$ and a vector $v \in T_yY$, we may adjust F in a small neighborhood of x so that the map remains C^{k+1} and its derivative hits v.

We now conclude that $\{F \in A | \rho_a \pitchfork W_i\}$ is residual in A, for each stratum W_i. Since the intersection of countably many residual sets is (by definition) residual, we in fact see that

$$\{F \in A \mid \rho_F \pitchfork W_i \;\; \forall\, i \in I, \}$$

is residual, hence dense, in A. In particular, since A is non-empty, there exists a map $F : S^k \to Y$ such that $F|_{S^{k-1}} = f$ and $\rho_F = F + H$ is transverse to each W_i. Since $k < \mu = \mathrm{codim}(W_i)$, this implies that the image of $F + H$ must be disjoint from each W_i. Hence $(F + H)(S^k) \subset U$, and so f represents the zero element in $\pi_{k-1}U$. $\square$

We can now prove the main result of this section. This result extends work of Ho and Liu, who showed that spaces of flat connections over surfaces are connected [23, Theorem 20]. We note, though, that their work applies to general structure groups G.

Proposition 4.3.7 *Let $M = M^g$ denote a compact Riemann surface of genus g, and let $n > 1$ be an integer. Then the space $\mathcal{A}^k_{\mathrm{flat}}(n)$ of flat connections on a trivial rank n bundle over M is $2g(n-1)$-connected, and if M is a non-orientable surface with double cover M^g, then the space of flat connections on any principal $U(n)$-bundle over M is $(g(n-1)-1)$-connected.*

Proof. We begin by noting that Sobolev spaces (of sections of fiber bundles) over compact manifolds are always second countable; this follows from Bernstein's proof of the Weierstrass theorem since we may approximate any function by smooth functions, and locally we may approximate smooth functions (uniformly up to the kth derivative for any k) by Bernstein polynomials.

I. The orientable case: By Theorem 4.2.14, we know that the inclusion

$$\mathcal{A}^k_{\mathrm{flat}}(n) \hookrightarrow \mathcal{C}^k_{ss}(n)$$

is a homotopy equivalence, so it suffices to show that $\mathcal{C}^k_{ss}(n)$ is $2g(n-1)$-connected. Since $\mathcal{C}^k(n)$ is a second countable Banach space, we may apply Corollary 4.3.6. The Harder-Narasimhan stratification gives the desired decomposition of $\mathcal{C}^k(n) - \mathcal{C}^k_{ss}(n)$ into locally closed submanifolds of finite codimension. The result now follows from the calculation of codimensions in Lemma 4.3.3.

II. The non-orientable case: We work in the set-up of non-orientable Yang-Mills theory, as developed by Ho and Liu [23]. Let M be a non-orientable surface with double cover M^g, and let P be a principal $U(n)$-bundle over M. Let $\pi : M^g \to M$ be the projection, and let $\widetilde{P} = \pi^* P$. Then the deck transformation $\tau : M^g \to M^g$ induces an involution $\widetilde{\tau} : \widetilde{P} \to \widetilde{P}$, and $\widetilde{\tau}$ acts on the space $\mathcal{A}^k(\widetilde{P})$ by pullback. Connections on P pull back to connections on $\widetilde{P}$, and in fact, as observed by Ho [22], the image of the pullback map is precisely the set of fixed points of τ. Hence we have a homeomorphism $\mathcal{A}^k(P) \cong \mathcal{A}^k(\widetilde{P})^{\widetilde{\tau}}$, which we treat as an identification. The Yang-Mills functional L is invariant under $\widetilde{\tau}$, and hence its gradient flow restricts to a flow on $\mathcal{A}^k(P)$.

The flat connections on P pull back to flat connections on $\widetilde{P}$, and again the image of $\mathcal{A}^k_{\text{flat}}(P)$ in $\mathcal{A}(\widetilde{P})$ is precisely $\mathcal{A}^k_{\text{flat}}(\widetilde{P})^{\widetilde{\tau}}$. If we let $\mathcal{C}^k_{ss}(P)$ denote the fixed set $\mathcal{C}^k_{ss}(\widetilde{P})^{\widetilde{\tau}}$, then the gradient flow of L restricts to give a deformation retraction from $\mathcal{C}^k_{ss}(P)$ to $\mathcal{A}^k_{\text{flat}}(P)$. The complement of $\mathcal{C}^k_{ss}(P)$ in $\mathcal{A}^k(P)$ may be stratified as follows: for each Harder-Narasimhan stratum $\mathcal{C}^k_{\mu}(\widetilde{P}) \subset \mathcal{A}^k(\widetilde{P}) \cong \mathcal{C}^k\left(\widetilde{P} \times_{U(n)} \mathbb{C}^n\right)$, we consider the fixed set $\mathcal{C}^k_{\mu}(P) := \left(\mathcal{C}^k_{\mu}(\widetilde{P})\right)^{\widetilde{\tau}}$. By [23, Proposition 17], $\mathcal{C}^k_{\mu}(P)$ is a real submanifold of $\mathcal{A}^k(P)$, and if it is non-empty then its real codimension in $\mathcal{A}^k(P)$ is half the real codimension of $\mathcal{C}^k_{\mu}(\widetilde{P})$ in $\mathcal{A}^k(\widetilde{P})$. The codimensions of the non semi-stable strata $\mathcal{C}^k_{\mu}(P)$ are hence at least $g(n-1)+1$ (by Lemma 4.3.3). The result now follows from Corollary 4.3.6. □

Corollary 4.3.8 *For any compact Riemann surface M and any $n \geqslant 1$, the representation space* $\mathrm{Hom}(\pi_1(M), U(n))$ *is connected. In particular,* $\mathrm{Rep}(\pi_1 M)$ *is stably group-like.*

Proof. The genus 0 case is, of course trivial. Now, for any $g, n \geqslant 1$ we have $2g(n-1) \geqslant 0$, so Proposition 4.3.7 implies that $\mathcal{A}^k_{\text{flat}}(n)$ is connected. Connectivity of $\mathrm{Hom}(\pi_1(M), U(n))$ will follow from Proposition 4.2.8, once we check that any $U(n)$ bundle over a Riemann surface which admits a flat connection is trivial.

Let P be a $U(n)$-bundle with a flat connection. Then the Chern classes of P are all zero; this follows from Chern-Weil theory, but can also be seen more elementarily as follows. Since M is 2-dimensional, only c_1 can be non-zero and $c_1(P) = c_1(\det P)$. Now, P is isomorphic to the bundle E_ρ associated to its holonomy representation ρ via mixing with universal cover, and $\det(E_\rho) \cong E_{\det\rho}$. But $\det(\rho)$ is a homomorphism to the abelian group $U(1)$, and hence factors through $H_1(M) = \mathbb{Z}^{2g}$; since $\mathbb{Z}^{2g}$ is free abelian we may now connect this homomorphism to the identity by a path. This gives a bundle homotopy between $\det(E_\rho)$ and the trivial bundle. (I learned this argument from [45].)

Now, the bundle P is determined up to isomorphism by its classifying map $M \to BU(n)$, and since the 3-skeleton of (a minimal cell complex for) $BU(n)$ is a 2-sphere, we see that this map is classified by its degree. But this degree is precisely the Chern class of P, and hence must be nullhomotopic. □

Corollary 4.3.9 *Let M be a compact, non-orientable, aspherical surface. Then for any $n \geqslant 1$, the representation space $\mathrm{Hom}(\pi_1 M, U(n))$ has two connected components, and if $\rho \in \mathrm{Hom}(\pi_1 M, U(n))$ and $\psi \in \mathrm{Hom}(\pi_1 M, U(m))$ lie in the non-identity components, then $\rho \oplus \psi$ lies in the identity component of $\mathrm{Hom}(\pi_1 M, U(n+m))$. In particular, $\mathrm{Rep}(\pi_1 M)$ is stably group-like.*

Proof. It follows immediately from Proposition 4.3.7 that the space of flat connections on any principal $U(n)$-bundle over M is connected, unless $n = 2$ and the genus of the universal cover of M is 1, i.e. M is the Klein bottle. In this case of the Klein bottle, we give a separate algebraic argument in Proposition 6.1.11. (Note though that our main interest at the moment is in the statement that $\mathrm{Rep}(\pi_1 M)$ is stably group-like, and to prove this, we are free to ignore the structure of $\mathrm{Hom}(\pi_1 M, U(n))$ for small n.)

Now, we claim that there are precisely two isomorphism types of principal $U(n)$-bundles over any non-orientable surface. A map from M into $BU(n)$ may be homotoped to a cellular map, and since the 3-skeleton of $BU(n)$ is a 2-sphere, the classification of $U(n)$-bundles is independent of n. Hence it suffices to compute relative K-theory, which is easily done using the Mayer-Vietoris sequence together with the fact that M is stably a wedge (see Lemma 4.5.1). (Note here that since $BU(n)$ is simply connected, the sets of unbased and based homotopy classes of maps $M \to BU(n)$ are the same.) The homeomorphism from Proposition 4.2.8 now shows that $\mathrm{Hom}(\pi_1 M, U(n))$ has at most two components, and the simple invariant developed by Ho and Liu [24, 25] easily shows that the representation spaces have at least two components (a discussion of their invariant, adapted to the present situation, appears in Chapter 6; see in particular Proposition 6.1.5 and the end of the proof of Theorem 6.1.9).

To complete the proof, we need to show that the sum of two representations in the non-identity components of $\mathrm{Hom}(\pi_1 M, U(-))$ lies in the identity component. It suffices to check that the sum of two non-trivial bundles P and Q over M is trivial. The first Chern classes satisfy $c_1(P \oplus Q) = c_1(P) \oplus c_1(P)$, and the first Chern class of a non-trivial bundle on M is always the non-trivial element of $H^2(M, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ (certainly there exists a $U(n)$ bundle over M with non-zero first Chern class, and there is only one non-trivial bundle in each dimension). So $c_1(P \oplus Q) = 0$, and since, as just explained, all non-trivial bundles over M have non-trivial first Chern class, we conclude that $P \oplus Q$ is trivial. □

4.4 Proof of the Atiyah-Segal theorem for surface groups

Theorem 4.4.1 *Let M be a compact, aspherical surface (in other words, $M \neq S^2, \mathbb{R}P^2$). Then for $* > 0$,*

$$K^*_{\mathrm{def}}(\pi_1(M)) \cong K^*(M),$$

where $K^*(M) = \pi_* \mathrm{Map}(M, \mathbb{Z} \times BU)$ denotes the complex K-theory of M. In the non-orientable case, this in fact holds in degree 0 as well; in the orientable case, we have $K^0_{\mathrm{def}}(\pi_1(M)) \cong \mathbb{Z}$.

The cohomology of a compact surface is easily computed, and together with the Atiyah-Hirzebruch spectral sequence this immediately yields a computation of complex K-theory. Thus Theorem 4.4.1 gives a complete computation of the deformation K-groups.

Corollary 4.4.2 *If M^g is a compact Riemann surface of genus $g > 0$, then we have*

$$K^*_{\mathrm{def}}(\pi_1 M^g) = \begin{cases} \mathbb{Z}, & * = 0 \\ \mathbb{Z}^{2g}, & * \text{ odd} \\ \mathbb{Z}^2, & * \text{ even}, * > 0. \end{cases}$$

Let M be a compact, non-orientable surface of the form $M = M^g \# N_j$ ($g \geqslant 0$), where $j = 1$ or 2 and $N_1 = \mathbb{R}P^2$, $N_2 = \mathbb{R}P^2 \# \mathbb{R}P^2$ (so N_2 is the Klein bottle). So long as $M \neq \mathbb{R}P^2$, we have:

$$K^*_{\mathrm{def}}(\pi_1 M^g \# N_j) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & * \text{ even} \\ \mathbb{Z}^{2g+j-1}, & * \text{ odd.} \end{cases}$$

Proof of Theorem 4.4.1.

I. The orientable case: Let M be a Riemann surface of genus $g > 0$. We will show that for any $* \geq 0$,

$$K^*_{\mathrm{def}}(\pi_1(M)) \cong \pi_*(\mathrm{Map}_0(M, BU)),$$

where Map_0 denotes the connected component of the constant map. This will clearly suffice. In fact, we will exhibit a zig-zag of weak equivalences between the zeroth space of the deformation K-theory spectrum and the space $\mathbb{Z} \times \mathrm{Map}_0(M, BU)$.

By Proposition 4.1.1, the zeroth space of the spectrum $K_{\mathrm{def}}(\pi_1 M)$ is weakly equivalent to

$$\underset{\overrightarrow{\oplus 1}}{\mathrm{hocolim}}\, \mathrm{Rep}(\pi_1 M)_{hU}.$$

Corollary 4.2.12, together with the definition of $\mathrm{Rep}(\pi_1 M)_{hU}$, shows that this space is weakly equivalent to

$$\underset{\overrightarrow{\oplus \tau}}{\mathrm{hocolim}} \coprod_{[P^n]} \left(\mathcal{A}^k_{\mathrm{flat}}(P^n)\right)_{h\mathcal{G}^{k+1}(P^n)},$$

where the maps are induced by direct sum with the trivial connection τ on the trivial line bundle ε^1 (and the maps $\mathcal{G}^{k+1}(P^n) \to \mathcal{G}^{k+1}(P^n \oplus \varepsilon^1)$ are induced by block sum with the identity $1 \in \mathcal{G}^{k+1}(\varepsilon^1)$). Since $\mathcal{A}^k_{\text{flat}}(P^n) = \emptyset$ unless P^n is trivial (see the proof of Corollary 4.3.8), this space is simply

$$\mathbb{Z} \times \underset{\overrightarrow{\oplus\tau}}{\operatorname{hocolim}}\, \mathcal{A}^k_{\text{flat}}(\varepsilon^n)_{h\mathcal{G}_0^{k+1}(n)}.$$

Proposition 4.3.7 shows that the connectivity of the projections $\mathcal{A}^k_{\text{flat}}(\varepsilon^n)_{h\mathcal{G}^{k+1}(n)} \to B\mathcal{G}^{k+1}(n)$ tends to infinity, and hence these maps induce a weak equivalence

$$\mathbb{Z} \times \underset{n\to\infty}{\operatorname{hocolim}}\, \mathcal{A}^k_{\text{flat}}(\varepsilon^n)_{h\mathcal{G}^{k+1}(n)} \longrightarrow \mathbb{Z} \times \underset{n\to\infty}{\operatorname{hocolim}}\, B\mathcal{G}^{k+1}(n). \tag{4.3}$$

(Here we are simply using the fact that the homotopy groups of a mapping telescope are the colimit of the homotopy groups; one need not consider the homotopy fiber of this map.) By Lemma 4.2.3, the inclusion $\mathcal{G}^{k+1}(n) \hookrightarrow \mathcal{G}(n)$ is a weak equivalence, so we may replace $\mathcal{G}^{k+1}(n)$ with $\mathcal{G}(n)$ on the right. Recall that so far we have been using Milnor's functorial model for classifying spaces (see Remarks 4.1.2). To complete the proof, we will need to instead use Atiyah and Bott's model [6, Section 2] for the classifying space of $\mathcal{G}(n)$. Their result states that the natural map

$$\operatorname{Map}(M, EU(n)) \to \operatorname{Map}_0(M, BU(n))$$

is a universal principal $\operatorname{Map}(M, U(n)) = \mathcal{G}(n)$ bundle, where Map_0 denotes the connected component of the constant map. We now have weak equivalences (see Remark 4.1.2)

$$B\mathcal{G}(n) \longleftarrow \left(E\mathcal{G}(n) \times \operatorname{Map}(M, EU(n))\right)/\operatorname{Map}(M, U(n)) \longrightarrow \operatorname{Map}_0(M, BU(n)),$$

which are natural in n and hence induce weak equivalences on homotopy colimits (taken with respect to the maps induced by the standard inclusions $U(n) \hookrightarrow U(n+1)$). The space $\operatorname{hocolim}_{n\to\infty} \operatorname{Map}_0(M, BU(n))$ is weakly equivalent to the colimit $\operatorname{Map}_0(M, BU)$, since maps from compact sets into a colimit land in some finite piece.

Hence we have a zig-zag of weak equivalences connecting the zeroth space of $K_{\text{def}}(\pi_1 M)$ to $\mathbb{Z} \times \mathrm{Map}_0(M, BU)$, as desired.

II. The non-orientable case: Let M be a non-orientable surface. Once again, Proposition 4.1.1 and Corollary 4.2.12 tell us that the zeroth space of $K_{\text{def}}(\pi_1 M)$ is weakly equivalent to

$$\underset{\overrightarrow{\oplus\tau}}{\mathrm{hocolim}} \coprod_{[P^n]} \left(\mathcal{A}^k_{\text{flat}}(P^n)\right)_{h\mathcal{G}^{k+1}(P^n)}.$$

By Proposition 4.3.7 we know that $\mathcal{A}^k_{\text{flat}}(P^n)$ is $\left(g\left(\widetilde{M}\right)(n-1)-1\right)$-connected for any $U(n)$-bundle P^n, where $g\left(\widetilde{M}\right)$ denotes the genus of the double cover of M. Since we have assumed $M \neq \mathbb{R}P^2$, we know that $g\left(\widetilde{M}\right) > 0$, and hence the connectivity of $\mathcal{A}^k_{\text{flat}}(P^n)$ tends to infinity with n. This shows that the natural map

$$\underset{\overrightarrow{\oplus\tau}}{\mathrm{hocolim}} \coprod_{[P^n]} \left(\mathcal{A}^k_{\text{flat}}(P^n)\right)_{h\mathcal{G}^{k+1}(P^n)} \longrightarrow \underset{\overrightarrow{\oplus 1}}{\mathrm{hocolim}} \coprod_{[P^n]} B\mathcal{G}^{k+1}(P^n)$$

is a weak equivalence (on the right hand side, 1 denotes the identity element in $\mathcal{G}^{k+1}(\varepsilon^1)$). As in the orientable case, we may now switch to the Atiyah-Bott models for $B\mathcal{G}(P^n)$, obtaining the space

$$\underset{n\to\infty}{\mathrm{hocolim}} \coprod_{[P^n]} \mathrm{Map}^{P^n}(M, BU(n)),$$

where Map^{P^n} denotes the component of the mapping space consisting of those maps $f : M \to BU(n)$ with $f^*(EU(n))$ isomorphic to P^n. But since the union is taken over *all* isomorphism classes, this space is simply

$$\mathbb{Z} \times \underset{n\to\infty}{\mathrm{hocolim}}\, \mathrm{Map}(M, BU(n)),$$

and as before is weakly equivalent to

$$\mathbb{Z} \times \mathrm{Map}(M, BU) = \mathrm{Map}(M, \mathbb{Z} \times BU).$$

□

We now make the following conjecture regarding the homotopy type of the spectrum $K_{\mathrm{def}}(\pi_1 M)$, as a module over the connective K-theory spectrum **ku**. Note that it is easy to check (using Theorem 4.4.1) that the homotopy groups of the proposed spectrum are the same as $K^*_{\mathrm{def}}(\pi_1(M))$.

Conjecture 4.4.3 *For any Riemann surface M^g, the spectrum $K_{\mathrm{def}}(\pi_1 M)$ is weakly equivalent, as a* **ku***-module, to*

$$\mathbf{ku} \vee \left(\bigvee_{2g} \Sigma \mathbf{ku} \right) \vee \Sigma^2 \mathbf{ku}.$$

4.5 Stable representation spaces

Using an argument similar to the proof of Theorem 4.4.1, we can extract the homotopy type of the representation spaces $\mathrm{Hom}(\pi_1(M), U(n))$, after stabilizing with respect to rank, as well as the connectivity of the inclusions

$$\mathrm{Hom}(\pi_1(M), U(n)) \hookrightarrow \mathrm{Hom}(\pi_1(M), U(n+1)).$$

First we need a simple lemma regarding the (stable) homotopy types of surfaces, and a well-known fact about connectivity of mapping spaces.

Lemma 4.5.1 *Let M^g_j denote the surface $M^g \# N_j$, $j = 0, 1,$ or 2, where $N_0 = S^2$, $N_1 = \mathbb{R}P^2$, and $N_2 = K$ is the Klein bottle. Then*

$$\Sigma M^g_j \simeq \Sigma(\bigvee_{2g} S^1 \vee N_j).$$

Proof. In the standard cell decomposition of M^g_j, the single 2-cell is attached via the relator map

$$\alpha_j = \prod_1^g [a_i, b_i] \cdot \omega_j : S^1 \to \bigvee_{2g+j} S^1,$$

where

$$\omega_j = \begin{cases} *, & j = 0 \\ c^2, & j = 1 \\ cdc^{-1}d, & j = 2. \end{cases}$$

Here a_i, b_i, c and d denote the identity maps from S^1 to the various wedge factors in $\bigvee_{2g+j} S^1$ and $*$ denotes the constant map to the wedge point. Hence we may identify M_j^g with the mapping cone

$$C(\alpha_j) = C(S^1) \cup_{\alpha_j} \bigvee_{2g+j} S^1.$$

Letting $C'(X \to Y) = C(X \to Y)/\{x_0\} \times I$ denote the reduced mapping cone, where $x_0 \in X$ is the basepoint, it is easy to check that reduced mapping cones commute with reduced suspensions in the sense that there is a homeomorphism

$$C'(\Sigma X \xrightarrow{\Sigma f} \Sigma Y) \cong \Sigma C'(X \xrightarrow{f} Y).$$

Hence $\Sigma M = \Sigma(C\alpha_j) \simeq C(\Sigma\alpha_j)$, and since Σ induces a homomorphism

$$\pi_1 \left(\bigvee_{2g+j} S^1 \right) \longrightarrow \pi_2 \left(\sum \left(\bigvee_{2g+j} S^1 \right) \right),$$

we have $\Sigma\alpha_j = \Sigma(\prod_1^g [a_i, b_i] \cdot \omega_j) \simeq \prod_1^g [\Sigma a_i, \Sigma b_i] \cdot \Sigma\omega_j$. But π_2 is abelian, so for each i, $[a_i, b_i]$ is nullhomotopic and hence $C(\Sigma\alpha_j)$ is homotopy equivalent to the mapping cone of the map

$$\Sigma\omega_j : S^2 \longrightarrow \sum \left(\bigvee_{2g+j} S^1 \right) \cong \bigvee_{2g+j} S^2.$$

Letting $\overline{\omega}_j$ denote the map $S^1 \to \bigvee_j S^1$ defined by the same word ω_j, we now have

$$C(\Sigma\omega_j) = \left(\bigvee_{2g} S^2 \right) \vee \Sigma(C\overline{\omega}_j) = \left(\bigvee_{2g} S^2 \right) \vee \Sigma N_j = \Sigma \left(\bigvee_{2g} S^1 \vee N_j \right),$$

as desired. $\square$

Lemma 4.5.2 *Let $f : X \to Y$ be an d-connected map. Then for any finite k-dimensional CW-complex K, the map*

$$Map_*(K, X) \longrightarrow Map_*(K, Y)$$

induced by f is $(d-k)$-connected. The same statement holds for unbased mapping spaces. (Here an n-connected map is a map inducing isomorphisms on homotopy through dimension n, and inducing a surjection on π_{n+1}. When the spaces in question are not connected, we require these conditions for all *compatible choices of basepoints.)*

Proof. We prove the statement in the based case; the argument in the unbased case is identical. The proof is by induction on k. If $k = 0$, then K is a finite set of points, and so $\mathrm{Map}_*(K, X)$ and $\mathrm{Map}_*(K, Y)$ are just products of $|K|$ copies of X and Y respectively. So the result is immediate in this case. Now, assume the result for spaces of dimension less that k. Letting $K^{(k-1)}$ denote the $(k-1)$-skeleton, the inclusion $K^{(k-1)} \hookrightarrow K$ is a cofibration and hence the restriction map

$$\mathrm{Map}_*(K, X) \longrightarrow \mathrm{Map}_*(K^{(k-1)}, X)$$

is a fibration, and similarly for Y. The fibers over the constant map are then the based mapping spaces $\mathrm{Map}_*(K/K^{(k-1)}, X)$ and $\mathrm{Map}_*(K/K^{(k-1)}, Y)$. Since $K/K^{(k-1)}$ is a wedge of k-spheres, we find that $\mathrm{Map}_*(K/K^{(k-1)}, X)$ is a product of copies of $\Omega^k X$ and similarly for Y. Since $f : X \to Y$ is d-connected, the map $\Omega^k X \to \Omega^k Y$ is $(d-k)$-connected, and by induction we may assume that $\mathrm{Map}_*(K^{(k-1)}, X) \to \mathrm{Map}_*(K^{(k-1)}, Y)$ is $d - (k-1) = d - k + 1$ connected. The result now follows (using the strong 5-lemma) from the diagram of fibrations

$$\begin{tikzcd}
(\Omega^k X)^n \arrow[r] \arrow[d] & (\Omega^k Y)^n \arrow[d] \\
\mathrm{Map}_*(K, X) \arrow[r] \arrow[d] & \mathrm{Map}_*(K, Y) \arrow[d] \\
\mathrm{Map}_*(K^{(k-1)}, X) \arrow[r] & \mathrm{Map}_*(K^{(k-1)}, Y).
\end{tikzcd}$$

□

Theorem 4.5.3 *Let $M = M^g$ be a compact Riemann surface of genus $g > 0$. Then there is a homotopy equivalence*

$$\mathrm{Hom}(\pi_1(M^g), U) \simeq U^{2g} \times BU,$$

and the inclusions

$$\mathrm{Hom}(\pi_1(M^g), U(n)) \xhookrightarrow{i_n} \mathrm{Hom}(\pi_1(M^g), U(n+1))$$

are precisely $(2n-2)$-connected. Hence for $k \leqslant 2n-2$, we have

$$\pi_k \mathrm{Hom}(\pi_1 M, U(n)) \cong \pi_k \left(U^{2g} \times BU\right) = \begin{cases} 0, & k = 0, \\ \mathbb{Z}^{2g}, & k \text{ odd}, \\ \mathbb{Z}, & k = 2l \text{ with } l > 0. \end{cases}$$

The phrase "precisely $(2n-2)$-connected" means that the maps i_n are not $(2n-1)$-connected: we will show that these maps do not induce surjections on π_{2n}. One may also ask whether i_n induces a isomorphism on π_{2n-1}, rather than just a surjection. In general we do not know the answer, but when $n = 1$, $U(1) = S^1$ is abelian and we have

$$\mathrm{Hom}(\pi_1(M^g), S^1) \cong \mathrm{Hom}(\mathbb{Z}^{2g}, S^1) \cong (S^1)^{2g}.$$

Hence the surjections

$$\mathbb{Z}^{2g} = \pi_1 \left(\mathrm{Hom}(\pi_1(M^g), U(1)\right) \longrightarrow \pi_1 \left(\mathrm{Hom}(\pi_1(M^g), U(m))\right) = \mathbb{Z}^{2g}$$

must in fact be isomorphisms (note that on the right hand side, this group is in the stable range so long as $m \geqslant 2$). (One can also see this from the proof of Theorem 4.5.3, using the fact that all holomorphic structures on $M \times \mathbb{C}$ are semi-stable, so $\mathcal{A}^k_{\text{flat}}(\varepsilon^1)$ is contractible.) The determinant maps $U(m) \to U(1)$ split the inclusions $U(1) \hookrightarrow U(m)$, and hence induce inverse isomorphisms on fundamental groups of the representation spaces.

Corollary 4.5.4 *Let M^g be a Riemann surface of genus $g > 1$. Then the maps*

$$\mathbb{Z}^{2g} = \pi_1\left(\mathrm{Hom}(\pi_1(M^g), U(1))\right) \longrightarrow \pi_1\left(\mathrm{Hom}(\pi_1(M^g), U(m))\right)$$

are isomorphisms, with inverses induced by the determinant maps $U(m) \to U(1)$.

Proof of Theorem 4.5.3. In analogy with the proof of Theorem 4.4.1, we have a diagram

$$\begin{tikzcd}
\mathrm{Hom}(\pi_1(M), U(n)) \arrow[r, "i_n"] & \mathrm{Hom}(\pi_1(M), U(n+1)) \\
\mathcal{A}^k_{\mathrm{flat}}(\varepsilon^n)_{h\mathcal{G}_0^{k+1}(n)} \arrow[r] \arrow[u, "\wr"'] \arrow[d, "p_n"] & \mathcal{A}^k_{\mathrm{flat}}(\varepsilon^{n+1})_{h\mathcal{G}_0^{k+1}(n+1)} \arrow[u, "\wr"] \arrow[d, "p_{n+1}"] \\
B\mathcal{G}_0^{k+1}(n) \arrow[r] \arrow[d, leftrightarrow, "\wr"'] & B\mathcal{G}_0^{k+1}(n+1) \arrow[d, leftrightarrow, "\wr"] \\
\mathrm{Map}^0_*(M, BU(n)) \arrow[r] & \mathrm{Map}^0_*(M, BU(n+1)).
\end{tikzcd} \tag{4.4}$$

The double headed arrows in the bottom square abbreviate a zig-zag of weak equivalences, as we will explain. The notation Map^0_* denotes the component of the based mapping space consisting of all null-homotopic maps. The map

$$\mathrm{Map}_*(M, EU(n)) \to \mathrm{Map}^0_*(M, BU(n))$$

is a universal principal $\mathcal{G}_0(n)$-bundle (the proof is analogous to the corresponding proof for $\mathcal{G}(n)$, given in [6, Section 2]) so we have weak equivalences

$$B\mathcal{G}_0^{k+1}(n) \xrightarrow{\simeq} B\mathcal{G}_0(n) \simeq \mathrm{Map}^0_*(M, BU(n)).$$

The final weak equivalence is again a natural zig-zag, just as in Remark 4.1.2.

In Diagram (4.4), the first and third vertical maps are weak equivalences. Since $\mathcal{A}^k_{\mathrm{flat}}(\varepsilon^n)$ has connectivity $2g(n-1)$, the fibration

$$p_n : \mathcal{A}^k_{\mathrm{flat}}(\varepsilon^n)_{h\mathcal{G}_0^{k+1}(n)} \longrightarrow B\mathcal{G}_0^{k+1}(n)$$

is a $2g(n-1)$-connected map and similarly p_{n+1} is $(2gn)$-connected; note that since

$g \geqslant 1$ both of these connectivities are at least $2n-2$.

The inclusions $U(n) \hookrightarrow U(n+1)$ are $(2n-1)$-connected (this follows from the fibration $U(n) \hookrightarrow U(n+1) \longrightarrow S^{2n+1}$) so by Lemma 4.5.2, the inclusions

$$\mathrm{Map}^0_*(M, BU(n)) \hookrightarrow \mathrm{Map}^0_*(M, BU(n+1))$$

are $(2n-2)$-connected. It now follows from commutativity of Diagram (4.4) that the inclusions

$$i_n : \mathrm{Hom}(\pi_1 M, U(n)) \hookrightarrow \mathrm{Hom}(\pi_1 M, U(n+1))$$

are $(2n-2)$-connected (note that on π_{2n-1}, the map p_n induces a surjection and p_{n+1} induces an isomorphism, so we may conclude, as desired, that i_n induces a surjection on π_{2n-1}).

To show that the map i_n is not more highly connected, we will check that the induced map

$$\pi_{2n}\mathrm{Hom}(\pi_1 M, U(n)) \longrightarrow \pi_{2n}\mathrm{Hom}(\pi_1 M, U(n+1))$$

is *never* surjective. We first note that surjectivity of this map would imply surjectivity of the other horizontal maps in Diagram (4.4); note that the connectivity of p_{n+1} is (at least) $2gn \geqslant 2n$. So we just need to show that the bottom map

$$\pi_{2n}\mathrm{Map}^0_*(M, BU(n)) \longrightarrow \pi_{2n}\mathrm{Map}^0_*(M, BU(n+1)) \cong \mathbb{Z}$$

is not surjective. The group on the left does not lie in the stable range, and we now show that it is in fact torsion. The skeletal filtration of M induces a long exact sequence in homotopy for the spaces of based maps into $BU(n)$, and this sequence has the form

$$\cdots \longrightarrow \pi_{2n}\Omega^2 BU(n) \longrightarrow \pi_{2n}\mathrm{Map}_*(M, BU(n)) \longrightarrow \pi_{2n}\left(\Omega BU(n)\right)^{2g} \longrightarrow \cdots.$$

The left and right terms are $\pi_{2n+1}U(n)$ and $\left(\pi_{2n}U(n)\right)^{2g}$, and these groups are both torsion. In fact, $\pi_{2n}U(n) \cong \mathbb{Z}/n!\mathbb{Z}$ [9], and $\pi_{2n+1}U(n)$ is trivial for odd n, and isomorphic to $\mathbb{Z}/2$ for even n [46, 27]. This completes our connectivity calculations.

To determine the homotopy type of $\mathrm{Hom}(\pi_1(M^g), U)$, we begin by noting the homeomorphism

$$\operatorname*{colim}_{n\to\infty} \mathrm{Hom}(\pi_1(M^g), U(n)) \xrightarrow{\cong} \mathrm{Hom}(\pi_1(M^g), U).$$

Moreover, the map from the homotopy colimit to the colimit is a weak equivalence because in either case compact sets lie in some finite piece. So we have

$$\mathrm{Hom}(\pi_1(M^g), U) \simeq \operatorname*{hocolim}_{n\to\infty} \mathrm{Hom}(\pi_1(M^g), U(n)).$$

Now, taking homotopy colimits in Diagram (4.4) gives

$$\operatorname*{hocolim}_{n\to\infty} \mathrm{Hom}(\pi_1(M^g), U(n)) \simeq \operatorname*{hocolim}_{n\to\infty} \mathrm{Map}_*^0(M, BU(n)).$$

Furthermore, the right-hand side is again weakly equivalent to the colimit

$$\operatorname*{colim}_{n\to\infty} \mathrm{Map}_*^0(M, BU(n)) \cong \mathrm{Map}_*^0(M, BU).$$

Bott periodicity, adjointness, and Lemma 4.5.1 combine to give a sequence of homotopy equivalences

$$\mathrm{Map}_*^0(M^g, BU) = \mathrm{Map}_*^0(M^g, \mathbb{Z}\times BU) \simeq \mathrm{Map}_*^0(M^g, \Omega U) \cong \mathrm{Map}_*^0(\Sigma M^g, U)$$

$$\simeq \mathrm{Map}_*^0(\Sigma(S^2 \vee \bigvee_{2g} S^1), U) \simeq \mathrm{Map}_*^0(S^2 \vee \bigvee_{2g} S^1, \Omega U) \simeq \mathrm{Map}_*^0(S^2 \vee \bigvee_{2g} S^1, \mathbb{Z}\times BU).$$

By Bott Periodicity, the full based mapping space on the right is homotopy equivalent to $\mathbb{Z} \times U^{2g} \times BU$. Hence we have a zig-zag of weak equivalences between $\mathrm{Hom}(\pi_1(M^g), U)$ and $U^{2g} \times BU$.

We claim that both of these spaces have the homotopy type of CW-complexes. Note that $\mathrm{Hom}(\pi_1(M^g), U(n))$ is a real algebraic variety and $\mathrm{Hom}(\pi_1(M^g), U(n-1))$ is included as a subvariety, and hence these spaces have compatible triangulations [21]. Thus the inclusions are cofibrations, meaning that the homotopy colimit of these spaces is a CW-complex and the projection to the colimit is a homotopy equivalence. Similarly, $U = \operatorname{colim} U(n)$ and $BU = \operatorname{colim} BU(n)$ have the homotopy type of (countable) CW-complexes; hence so does $U^{2g} \times BU$.

By using CW-approximations, we obtain a new zig-zag of weak equivalences between $\mathrm{Hom}(\pi_1(M^g), U)$ and $U^{2g} \times BU$, in which all other spaces are CW-complexes. By the Whitehead Theorem, we may (homotopy) invert the maps in this zig-zag to obtain an honest homotopy equivalence $\mathrm{Hom}(\pi_1(M^g), U) \xrightarrow{\simeq} U^{2g} \times BU$. □

There are similar results (proven in essentially the same manner) regarding the homotopy orbit spaces $\mathrm{Hom}(\pi_1(M^g), U(n))_{hU(n)}$. In particular, their colimit has the (weak) homotopy type of $U^{2g} \times (BU)^2$. Results of Park and Suh [38] show that $\mathrm{Hom}(\pi_1(M^g), U(n))$ is a $U(n)$-CW complex, and hence these homotopy orbit spaces are CW complexes as well. Moreover, the inclusions are cofibrations, and we may again conclude that we have an actual homotopy equivalence.

It would of course be interesting to have an explicit homotopy equivalences between these spaces, both for the representation spaces and their homotopy orbit spaces. The proofs clearly do not provide explicit maps.

Next we consider the case of a non-orientable surface $M^g_j = M^g \# N_j$, where $j = 1$ or 2. Recall that $N_1 = \mathbb{R}P^2$, and N_2 is the Klein bottle.

Theorem 4.5.5 *Let $M = M^g_j$ be as above. Then there is a homotopy equivalence*

$$\mathrm{Hom}(\pi_1(M^g_j), U) \simeq \mathbb{Z}/2\mathbb{Z} \times U^{2g} \times \mathrm{Map}_*(N_j, BU),$$

and letting

$$\mu_{n,g} = \begin{cases} 0, & n = 1, \\ \min\{2n-2, g(n-1)-1\}, & n > 1, \end{cases}$$

the inclusions

$$\mathrm{Hom}(\pi_1 M, U(n)) \hookrightarrow \mathrm{Hom}(\pi_1 M, U(n+1))$$

are $\mu_{n,g}$-connected. Hence in the stable range $k \leqslant \mu_{n,g}$, we have

$$\pi_k \mathrm{Hom}(\pi_1 M, U(n)) \cong \pi_k(\mathbb{Z}/2\mathbb{Z} \times U^{2g} \times \mathrm{Map}_*(N_j, BU))$$

$$= \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & k = 0 \\ \mathbb{Z}^{2g+j-1}, & k \text{ odd} \\ \mathbb{Z}/2\mathbb{Z}, & k = 2l \text{ with } l > 0. \end{cases}$$

The reader should recall our definition of connectivity from Lemma 4.5.2; in particular the above theorem is valid for any choice of basepoints.

As in the orientable case, several comments are in order. First, say $g > 2$ and $n \geqslant 2$. Then the connectivity bound in Theorem 4.5.5 is $2n - 2$, and one can see just as in the orientable case that the maps

$$\pi_{2n}\mathrm{Hom}(\pi_1 M, U(n)) \longrightarrow \pi_{2n}\mathrm{Hom}(\pi_1 M, U(n+1))$$

are not surjective. Hence in these cases the connectivity bound cannot be improved. When $n = 1$ (so $\mu_{n,g} = 0$), we have $\mathrm{Hom}(\pi_1(M^g_j, U(1)) \cong \mathbb{Z}/2\mathbb{Z} \times (S^1)^{2g+j-1}$, and as in the orientable case the maps

$$\mathrm{Hom}(\pi_1(M^g_j), U(1)) \longrightarrow \pi_1\mathrm{Hom}(\pi_1(M^g_j), U(m))$$

are all isomorphisms on π_1 (with inverse given by the determinant map), but are certainly not 1-connected. In the remaining cases, the upper bound for connectivity remains at $2n-2$, but the lower bound in Theorem 4.5.5 is either $n-2$ (when $g = 1$) or $2n - 3$ (when $g = 2$).

One can also show that the homotopy orbit spaces $\mathrm{Hom}(\pi_1(M^g_j), U)_{hU}$ have the homotopy type of $\mathbb{Z}/2\mathbb{Z} \times U^{2g} \times \mathrm{Map}_*(N_j, BU) \times BU$.

Proof of Theorem 4.5.5. The proof is essentially the same as in the orientable case. The only subtlety comes in checking connectivity of the map

$$\mathrm{Hom}(\pi_1 M^g, U(n)) \hookrightarrow \mathrm{Hom}(\pi_1 M^g, U(n+1)).$$

This is the same as the connectivity of the upper horizontal the map ι_n in the diagram

$$\begin{tikzcd} \coprod_{[P^n]} \mathcal{A}^k_{\text{flat}}(P)_{h\mathcal{G}_0^{k+1}(P)} \arrow[r, "\iota_n"] \arrow[d, "q_n"] & \coprod_{[P_{n+1}]} \mathcal{A}^k_{\text{flat}}(P_{n+1})_{h\mathcal{G}_0^{k+1}(P_{n+1})} \arrow[d, "q_{n+1}"] \\ \coprod_{[P^n]} B\mathcal{G}_0^{k+1}(P^n) \arrow[r, "j_n"] & \coprod_{[P_{n+1}]} B\mathcal{G}_0^{k+1}(P_{n+1}). \end{tikzcd} \tag{4.5}$$

When $k \leqslant \mu_{n,g}$, we know that q_n, q_{n+1} and j_n all induce isomorphisms on π_k, and hence so does ι_n (note that when $n = 1$, all holomorphic structures are semi-stable and hence q_1 is a weak equivalence). The fact that ι_n induces a surjection on $\pi_{1+\mu_{n,g}}$ follows easily from the fact that the connectivity of q_{n+1} is *more* than that of q_n.

The calculation of $\pi_*(\mathbb{Z}/2\mathbb{Z} \times U^{2g} \times \text{Map}_*(N_j, BU))$ follows from the long exact sequence for the (split) fibration

$$\text{Map}_*(N_j, BU)) \longrightarrow \text{Map}_*(N_j, BU)) \to BU,$$

together with knowledge of the groups $\pi_*\text{Map}(N_j, BU))$. After dimension 0 these are just the complex K-groups of N_j, and K-theory of $\mathbb{R}P^2$ (and of the Klein bottle) is easily calculated using the Mayer-Vietoris sequence. □

4.6 The stable coarse moduli space

We now turn to the quotient space $\text{Hom}(\pi_1 M, U)/U$, which we think of as the stable (coarse) moduli space of representations. Lawson has shown [30] that for any finitely generated group Γ, there is a homotopy cofiber sequence of spectra

$$\Sigma^2 K_{\text{def}}(\Gamma) \longrightarrow K_{\text{def}}(\Gamma) \longrightarrow R^{\text{def}}(\Gamma), \tag{4.6}$$

where $R^{\text{def}}(\Gamma)$ denotes the "deformation representation ring" of Γ, as we will explain. (The first map in this sequence is the Bott map in deformation K-theory, and is obtained from the Bott map in connective K-theory by smashing with $K_{\text{def}}(G)$.) We note that since the $K_{\text{def}}(\Gamma)$ is connective, the first two homotopy groups of $\Sigma^2 K_{\text{def}}(\Gamma)$ are zero, and hence the long exact sequence in homotopy associated to

(4.6) immediately gives isomorphisms

$$K^i_{\mathrm{def}}(\Gamma) \cong \pi_i R^{\mathrm{def}}(\Gamma) \tag{4.7}$$

for $i = 0, 1$.

The deformation representation ring $R^{\mathrm{def}}(\Gamma)$ is the spectrum associated to the abelian topological monoid

$$\overline{\mathrm{Rep}(\Gamma)} = \coprod_{n=0}^{\infty} \mathrm{Hom}(\Gamma, U(n))/U(n).$$

We briefly discuss the construction of this spectrum. Given any topological abelian monoid A (for which the inclusion of the identity is a cofibration), one may apply Segal's infinite loop space machine [42] to produce a connective Ω-spectrum; equivalently the bar construction BA is again an abelian topological monoid and one may iterate. In particular, the zeroth space of this spectrum is exactly ΩBA. Hence we have

$$\pi_* R^{\mathrm{def}}(\Gamma) \cong \pi_* \Omega B\left(\overline{\mathrm{Rep}(\Gamma)}\right).$$

It is in general rather easy to identify the group completion ΩBA when A is an abelian monoid; in light of Theorem 3.0.11 one essentially just needs an understanding of $\pi_0(A)$. In the case of surface groups, we have the following result, whose proof we sketch for completeness.

Proposition 4.6.1 *Let Γ be a finitely generated discrete group, and assume that* $\mathrm{Rep}(\Gamma)$ *stably group-like with respect to the trivial representation* $1 \in \mathrm{Hom}(\Gamma, U(1))$ *(e.g. $\Gamma = \pi_1 M$ with M an aspherical compact surface). Then the zeroth space of* $R^{\mathrm{def}}(\Gamma)$ *is weakly equivalent to* $K^0_{\mathrm{def}}(\Gamma) \times \mathrm{Hom}(\Gamma, U)/U$*. Hence we have*

$$\pi_* \mathrm{Hom}(\Gamma, U)/U \cong \pi_* R^{\mathrm{def}}(\Gamma)$$

for $* > 0$, and in particular $\pi_1 \mathrm{Hom}(\Gamma, U)/U \cong K^1_{\mathrm{def}}(\Gamma)$*.*

Proof. This follows from (a rather easy case of) Theorem 3.0.11. Specifically, when $\mathrm{Rep}(\Gamma)$ is stably group-like with respect to the trivial representation $1 \in$

$\mathrm{Hom}(\Gamma, U(1))$, the same is true for the monoid of isomorphism classes $\overline{\mathrm{Rep}(\Gamma)}$. We can now apply Theorem 3.0.11, because the additional hypothesis (that the representation 1 must be "anchored") is trivially satisfied for abelian monoids. Hence we conclude that there is a weak equivalence

$$\Omega B\left(\overline{\mathrm{Rep}(\Gamma)}\right) \simeq \underset{\overrightarrow{\oplus 1}}{\mathrm{hocolim}}\, \overline{\mathrm{Rep}(\Gamma)}.$$

The monoid of connected components of these spaces is the Grothendieck group of $\pi_0\left(\overline{\mathrm{Rep}(\Gamma)}\right)$, but since $U(n)$ is connected the projection induces an isomorphism

$$\pi_0 \mathrm{Rep}(\Gamma) \cong \pi_0(\overline{\mathrm{Rep}(\Gamma)}).$$

Thus the Grothendieck groups of these monoids are isomorphic, and by Lemma 2.0.5 the Grothendieck group on the left is $K^0_{\mathrm{def}}(\Gamma)$.

Since the space $\Omega B(\mathrm{Rep}(\Gamma))$ is a group-like H-space, all of its components are homotopy equivalent, and hence the same is true of the above homotopy colimit. To complete the proof we just need to check that one of these components, say

$$\underset{n\to\infty}{\mathrm{hocolim}}\, \mathrm{Hom}(\Gamma, U(n))/U(n),$$

is weakly equivalent to the actual colimit

$$\underset{n\to\infty}{\mathrm{colim}}\, \mathrm{Hom}(\Gamma, U(n))/U(n).$$

But the natural projection from the homotopy colimit to the colimit is a weak equivalence, because in each case compact sets land in some finite piece (for the colimit, this requires that points are closed in $\mathrm{Hom}(\Gamma, U(n))/U(n)$; this space is in fact Hausdorff because the orbits of $U(n)$ are compact, hence closed in $\mathrm{Hom}(\Gamma, U(n))$, which is a metric space, hence normal). $\square$

Combining Proposition 4.6.1 with (4.7) and the computation of $K_{\mathrm{def}}(\pi_1 M)$ in Theorem 4.4.1, we have:

Corollary 4.6.2 *If M^g is a compact Riemann surface of genus g, then*

$$\pi_1\left(\mathrm{Hom}(\pi_1 M, U)/U\right) \cong \mathbb{Z}^{2g}.$$

In the non-orientable cases, we have

$$\pi_1\left(\mathrm{Hom}(\pi_1 M^g \# \mathbb{R}P^2, U)/U\right) \cong \mathbb{Z}^{2g} \quad \text{and} \quad \pi_1\left(\mathrm{Hom}(\pi_1 M^g \# K, U)/U\right) \cong \mathbb{Z}^{2g+1},$$

where K denotes the Klein Bottle.

We note that the proof of this result requires not only the Yang-Mills theory used to prove Theorem 4.4.1 (which includes deep analytical results like Uhlenbeck compactness and convergence of the Yang-Mills flow) but also the modern stable homotopy theory underlying Lawson's cofiber sequence. His results require, for example, the model categories of module and algebra spectra constructed in [13].

Assuming Conjecture 4.4.3, we know that for Riemann surfaces, $K_{\mathrm{def}}(\pi_1 M^g)$ is free as a **ku**-module. Hence the Bott map is easily calculated, and one may compute the homotopy groups of $\mathrm{Hom}(\pi_1(M^g), U)/U$. It is interesting to note that they vanish above dimension 2. The reader should note the similarity between this calculation (and the previous theorem) and the main result of Lawson's paper [30], which states that U^k/U, the space of isomorphism classes of representations of a *free* group, has the homotopy type of $\mathrm{Sym}^\infty(S^1)^k = \mathrm{Sym}^\infty B(F_k)$. (Of course this space is homotopy equivalent to $(S^1)^k$.)

4.7 Free products

The behavior of deformation K-theory on free products will be described in Theorem 5.1.1. This theorem allows one to calculate $K^*_{\mathrm{def}}(\pi_1(M_1 \vee M_2))$ for any compact, aspherical surfaces M_1 and M_2 (using our computation of deformation K-theory of the factors in Theorem 4.4.1). Similarly, one may calculate $K^*(M_1 \vee M_2)$, obtaining the same answer (in the orientable case, we must assume $* > 0$). This provides, computationally at least, an Atiyah-Segal theorem for free products of surface groups. In this section we explain a more natural approach, using Yang-Mills theory.

We need some definitions and lemmas regarding gauge transformations and flat connections over wedges of manifolds. For these definitions, let $M_1, \ldots, M_l$ denote a collection of smooth, compact manifolds, and let $W = \bigvee_1^l M_i$. We let $\dim(W) = \max_i \dim(M_i)$.

It will be most convenient to work with trivial bundles only; this essentially amount to restricting attention to the identity component of $\mathrm{Hom}(\pi_1 W, U(n))$, which will suffice for computing $K^*_{\mathrm{def}}(\pi_1 W)$ (when $* > 0$ and $\dim(W) \leqslant 2$). We denote this component by $\mathrm{Hom}_I(-, U(n)$.

Definition 4.7.1 *We define the continuous gauge group of the trivial bundle $W \times \mathbb{C}^n$ by setting*

$$\mathcal{G}(W, n) = \mathrm{Map}(W, U(n)) \subset \prod_i \mathcal{G}(M_i, n),$$

and we define the L^2_{k+1} gauge group to be the subgroup $\mathcal{G}^{k+1}(W, n) \subset \prod_i \mathcal{G}^{k+1}(M_i, n)$ consisting of those maps which agree over the wedge point. Note that evaluation at the wedge point gives a surjective homomorphism $\mathcal{G}^{k+1}(W) \longrightarrow U(n)$, with kernel the product of the L^2_{k+1} based gauge groups of the M_i. We will denote this kernel by $\mathcal{G}^{k+1}_0(W, n)$, and refer to it as the based gauge group.

Remark 4.7.2 *We will assume throughout this section that our Sobolev parameter k is greater than $\dim(W)/2$. This will allow us to apply the results of Section 4.2 on each wedge factor of W.*

Just as in Lemma 4.2.3, the inclusion $\mathcal{G}^{k+1}(W, n) \hookrightarrow \mathcal{G}(W, n)$ is a weak equivalence, and just as in [6, Section 2] the space $\mathrm{Map}_0(W, BU(n))$ is a model for the classifying space of $\mathcal{G}(W, n)$.

Definition 4.7.3 *We define the space $\mathcal{A}^k(W, n)$ of L^2_k connections on the trivial bundle $(W) \times \mathbb{C}^n$ to be the product $\prod_i \mathcal{A}^k(M_i, n)$, and we define the space $\mathcal{A}^k_{\mathrm{flat}}(W, n)$ of flat L^2_k connections to be the subspace $\prod_i \mathcal{A}^k_{\mathrm{flat}}(M_i, n)$.*

Note that $\mathcal{A}^k(W, n)$ is always contractible, since it is a product of contractible (affine) spaces.

Lemma 4.7.4 *Assume* $\dim(W) \leqslant 2$*. The gauge group* $\mathcal{G}^{k+1}(W,n)$ *acts continuously on* $\mathcal{A}^k(W,n)$*, and the holonomy map induces a* $U(n)$*-equivariant homeomorphism*

$$\mathcal{A}^k_{\text{flat}}(W,n)/\mathcal{G}^{k+1}_0(W,n) \longrightarrow \text{Hom}_I(\pi_1 W, U(n)),$$

where the right-hand side denotes the component of the identity.

Proof. Continuity of the action follows from the fact that

$$\mathcal{G}^{k+1}(W,n) \subset \prod_i \mathcal{G}^{k+1}(M_i,n),$$

and the latter acts continuously on

$$\mathcal{A}^k_{\text{flat}}(W,n) = \prod_i \mathcal{A}^k_{\text{flat}}(M_i,n).$$

The desired homeomorphism now follows from Proposition 4.2.8, because the spaces and the groups in question are all products. Note here that over a one- or two-dimensional manifold, the space of flat connections on the trivial $U(n)$-bundle is connected (Proposition 4.3.7) so the holonomy of any flat connection on this bundle lies in $\text{Hom}_I(\pi_1 W, U(n))$. □

We can now prove the desired version of the Atiyah-Segal theorem for free products of surface groups. Note that there is a homotopy equivalence $B(G * H) \simeq BG \vee BH$; generally the classifying space of an injective amalgamated product is the homotopy pushout of the classifying spaces [20, Theorem 1B.11].

Theorem 4.7.5 *Let* $M_1, \ldots, M_k$ *be one- or two-dimensional aspherical manifolds, and let* $W = \bigvee_1^k M_i$*. Then there exist isomorphisms*

$$K^*_{\text{def}}(\pi_1(W)) \cong K^*(W)$$

for $* > 0$*. These isomorphisms are natural with respect to projections onto the wedge factors and inclusions of wedge factors. If all of the* M_i *are either circles or non-orientable surfaces, then the result holds for* $* = 0$ *as well.*

Proof. As in Proposition 4.1.1, we have an isomorphism

$$K^*_{\mathrm{def}}(\pi_1 W) \cong \pi_* \operatorname*{hocolim}_{\overrightarrow{\oplus 1}} \mathrm{Rep}(\pi_1 W)_{hU}.$$

For $* > 0$, the latter group is just

$$\pi_* \operatorname*{hocolim}_{n \to \infty} \mathrm{Hom}_I(\pi_1 W, U(n))_{hU(n)},$$

and one can easily check the result on π_0 separately. We can now proceed as in the proof of Theorem 4.4.1; naturality will be clear from the construction.

For each n, there is a zig-zag of maps connecting

$$\mathrm{Hom}_I(\pi_1(W), U(n))_{hU(n)} \text{ and } \mathrm{Map}_0(W, BU(n)),$$

and all but one of these maps is a weak equivalence. The remaining map is the projection

$$\mathcal{A}^k_{\mathrm{flat}}(W, n)_{h\mathcal{G}^{k+1}(W,n)} \xrightarrow{q_n} B\mathcal{G}^{k+1}(W, n).$$

Since all of these maps are natural in n, it will suffice to show that q_n becomes a weak equivalence after passing to homotopy colimits.

Since $\mathcal{A}^k(W, n)$ is the product of the $\mathcal{A}^k(M_i, n)$, we may stratify $\mathcal{A}^k(W, n)$ by the product of the stratifications on the factors. Moreover, each product stratum is still a locally closed submanifold of finite codimension, and in fact its codimension is just the sum of the codimensions of its factors. So the codimensions of the non-semi-stable strata still tend to infinity with n, and we may proceed just as in Proposition 4.3.7. □

Just as in Corollary 4.6.2, we have:

Corollary 4.7.6 *Let W be as in Theorem 4.7.5. Then there is an isomorphism*

$$\pi_1 \mathrm{Hom}(\pi_1 W, U)/U \cong K^1(W).$$

Chapter 5

Excision in deformation K-theory

We begin by describing the excision problem for amalgamated products. Let G, H, and K be finitely generated discrete groups, with homomorphisms $f_1 : K \to G$ and $f_2 : K \to H$. Then associated to the co-cartesian (i.e. pushout) diagram of groups

$$\begin{tikzcd} K \arrow[r, "f_1"] \arrow[d, "f_2"] & G \arrow[d] \\ H \arrow[r] & G *_K H \end{tikzcd}$$

there is a diagram of spectra

$$\begin{tikzcd} K_{\mathrm{def}}(G *_K H) \arrow[r] \arrow[d] & K_{\mathrm{def}}(G) \arrow[d, "f_1^*"] \\ K_{\mathrm{def}}(H) \arrow[r, "f_2^*"] & K_{\mathrm{def}}(K). \end{tikzcd} \tag{5.1}$$

We will say that the amalgamated product $G *_K H$ satisfies excision (for deformation K-theory) if diagram (5.1) is homotopy cartesian, i.e. if the natural map from $K_{\mathrm{def}}(G *_K H)$ to the homotopy pullback is a weak equivalence. Note that since we are dealing with connective Ω-spectra, this is the same as saying that the diagram of zeroth spaces is homotopy cartesian.

Excision may be thought of as the statement that deformation K-theory maps (certain) co-cartesian diagrams of groups to homotopy cartesian diagrams of spectra. Excision results are important from the point of view of computations, since associated to any homotopy cartesian diagram of spaces

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "h"] \\ Y \arrow[r, "k"] & Z \end{tikzcd}$$

there is a long exact "Mayer-Vietoris" sequence of homotopy groups

$$\ldots \longrightarrow \pi_k(W) \xrightarrow{f_* \oplus g_*} \pi_k(X) \oplus \pi_k(Y) \xrightarrow{h_* - k_*} \pi_k(Z) \xrightarrow{\partial} \pi_{k-1}(W) \longrightarrow \ldots \tag{5.2}$$

which comes from combining the long exact sequences associated to the vertical maps (see [20, p. 159]; note that the homotopy fibers of the vertical maps in a homotopy cartesian square are weakly equivalent). It is not difficult to check that if all the spaces involved are group-like H-spaces, and the maps are homomorphisms of H-spaces, then the maps in this sequence (including the boundary maps) are homomorphisms in dimension zero. Hence when applied to (the zeroth spaces of) the deformation K-theory in an amalgamation diagram, assuming excision one obtains a long exact sequence in K^*_{def}.

Deformation K-theory can fail to satisfy excision in low dimensions. We will now describe examples of this phenomenon (in the unitary case) that arise from the fundamental groups of Riemann surfaces. Letting $M = M^{g_1+g_2}$ denote the surface of genus $g_1 + g_2$ and F_k the free group on k generators, if we think of M as a connected sum then the Van Kampen Theorem gives us an amalgamation diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "c_1"] \arrow[d, "c_2"] & F_{2g_1} \arrow[d] \\ F_{2g_2} \arrow[r] & \pi_1(M). \end{tikzcd} \tag{5.3}$$

If we write the generators of F_{2g_i} as $a^i_1, b^i_1, \ldots, a^i_{g_i}, b^i_{g_i}$ the map c_i is the multiple-commutator map, sending $1 \in \mathbb{Z}$ to $\prod_{j=1}^{g_i}[a^i_j, b^i_j]$. If deformation K-theory were excisive on diagram (5.3), there would be a long exact sequence of the form (5.2).

Since the representation spaces of F_k are always connected, $\mathrm{Rep}(F_k)$ is stably group-like with respect to $1 \in \mathrm{Hom}(F_k, U(1))$. Hence (using Corollary 3.0.16) one finds that the induced map $c_i^* : K^*_{\mathrm{def}}(F_{2g_i}) \to K^*_{\mathrm{def}}(\mathbb{Z})$ may be identified with the map

$$\pi_*(\mathbb{Z} \times (U^{2g_i})_{hU}) \to \pi_*(\mathbb{Z} \times U_{hU})$$

induced by the multiple commutator map $C : U^{2g_i} \to U$ (here the actions of U are via conjugation). The induced map C_* on homotopy is always zero, and from the diagram of fibrations

$$\begin{tikzcd} U^{2g_i} \arrow[r] \arrow[d, "C"] & EU \times_U U^{2g_i} \arrow[r] \arrow[d] & BU \arrow[d, "||"] \\ U \arrow[r] & EU \times_U U \arrow[r] & BU \end{tikzcd}$$

one now concludes (using Bott Periodicity) that c_i^* is zero for $*$ odd.

Next, Lemma 2.0.5 implies that $K^0_{\mathrm{def}}(F_k) \cong \mathbb{Z}$ for any k, and moreover that any map between free groups induces an isomorphism on K^0_{def} (essentially K^0_{def} just records the dimension of a representation). Hence excision would give us a long exact sequence ending with

$$K^1_{\mathrm{def}}(F_{g_1}) \oplus K^1_{\mathrm{def}}(F_{g_2}) \xrightarrow{0} K^1_{\mathrm{def}}(\mathbb{Z}) \hookrightarrow K^0_{\mathrm{def}}(\pi_1(M)) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow \mathbb{Z},$$

which would imply an isomorphism $K^0_{\mathrm{def}}(\pi_1(M)) \cong \mathbb{Z} \oplus K^1_{\mathrm{def}}(\mathbb{Z})$. Now, $K^1_{\mathrm{def}}(\mathbb{Z}) \cong \pi_1(EU \times_U U)$, and it is well-known that the latter space is weakly equivalent to the free loop space of BU (see [18] for an elegant proof) and hence has fundamental group $\mathbb{Z}$. The representation spaces of the fundamental group of a compact Riemann surface are always connected (see Corollary 4.3.8), so as with F_k we have $K^0_{\mathrm{def}}(\pi_1(M)) \cong \mathbb{Z}$. This contradiction shows that excision cannot hold for such diagrams.

On the other hand, we do expect that connected sum decompositions satisfy excision above dimension zero. Theorem 4.4.1 tells us that the homotopy groups of $K_{\mathrm{def}}(\pi_1(M^{g_1} \# M^{g_2}))$ are isomorphic to $K^*(M^{g_1} \# M^{g_2})$ in positive dimensions, and the same is true for the free groups appearing in this amalgamation diagram

(Theorem 4.7.5). The Mayer-Vietoris sequences for the homotopy pullback $\mathcal{H}$ of the diagram

$$K_{\mathrm{def}}(F_{2g_1}) \longrightarrow K_{\mathrm{def}}(\mathbb{Z}) \longleftarrow K_{\mathrm{def}}(F_{2g_2})$$

and for $K^*(M^{g_1}\#M^{g_2})$ show that these groups are also (abstractly) isomorphic above dimension zero. Hence we conclude that the deformation K-groups of the amalgamated product are (abstractly) isomorphic to the homotopy groups of the homotopy pullback $\mathcal{H}$. Unfortunately, this is not enough to conclude that the natural map

$$K_{\mathrm{def}}\left(\pi_1(M^{g_1}\#M^{g_2})\right) \longrightarrow \mathcal{H}$$

is a weak equivalence; one needs some sort of naturality.

One approach is to try to relate the square

$$\begin{tikzcd}
K_{\mathrm{def}}\left(\pi_1(M^{g_1}\#M^{g_2})\right) \arrow[r] \arrow[d] & K_{\mathrm{def}}(F_{2g_2}) \arrow[d] \\
K_{\mathrm{def}}(F_{2g_1}) \arrow[r] & K_{\mathrm{def}}(\mathbb{Z})
\end{tikzcd}$$

to the square of mapping spaces

$$\begin{tikzcd}
\mathrm{Map}(M^{g_1}\#M^{g_2}, BU) \arrow[r] \arrow[d] & \mathrm{Map}\left((S^1)^{2g_2}, BU\right) \arrow[d] \\
\mathrm{Map}\left((S^1)^{2g_1}, BU\right) \arrow[r] & \mathrm{Map}(S^1, BU),
\end{tikzcd}$$

which is clearly homotopy cartesian (this is essentially excision for complex K-theory). The problem, though, is that the zig-zags of maps connecting the corners of these squares, as constructed in Chapter 4, does not complete to a diagram of squares: the multiple commutator maps $\mathbb{Z} \to F_{2g_i}$ do not correspond to maps from the space of flat connections over the appropriate wedges of circles to the space of flat connections over S^1 (flat connections over a wedge of manifolds were discussed in Section 4.7). There are other methods for relating deformation K-theory of a free group to the above mapping spaces, as in [30] or the appendix to [18], but these maps bear no obvious relation to our arguments for surface groups, so one again meets problems of commutativity. Nevertheless, we make the following conjecture:

Conjecture 5.0.7 *Let M_1 and M_2 be compact surfaces such that the connected sum $M_1 \# M_2$ is aspherical, and let k_i denote the rank of the free group $\pi_1(M_i - D^2)$, where $D^2 \subset M_i$ is a disk. Then the natural map*

$$K^*_{\mathrm{def}}(\pi_1(M_1 \# M_2)) \longrightarrow \pi_* \operatorname{holim}(K_{\mathrm{def}}(F_{k_1}) \longrightarrow \mathbb{Z} \longleftarrow K_{\mathrm{def}}(F_{k_2}))$$

is an isomorphism for $* > 0$. (If $M_1 \# M_2$ is non-orientable, then we expect that this holds for $* = 0$ as well; see Proposition 5.2.5 for further evidence.)

5.1 Excision for free products

In this section we present our results on the excision problem for free products and discuss some resulting computations. Using the results from Chapter 3, we will show that deformation K-theory satisfies excision for free products:

Theorem 5.1.1 *Let G and H be finitely generated discrete groups. Then the diagram of spectra*

$$\begin{tikzcd} K_{\mathrm{def}}(G * H) \arrow[r] \arrow[d] & K_{\mathrm{def}}(G) \arrow[d] \\ K_{\mathrm{def}}(H) \arrow[r] & K_{\mathrm{def}}(\{1\}) \end{tikzcd}$$

is homotopy cartesian.

Note that $K_{\mathrm{def}}(\{1\}) \simeq \mathbf{ku}$, the complex connective K-theory spectrum, since (in the unitary case) $\mathrm{Rep}(\{1\})_{hU} = \coprod_{n=0}^{\infty} BU(n)$. This also holds for general linear deformation K-theory, because $U(n) \simeq GL_n(\mathbb{C})$.

As discussed above, Theorem 5.1.1 yields a long exact sequence in K^*_{def}. In fact, the boundary maps in this sequence are always zero because the map $K^*_{\mathrm{def}}(G) \to K^*_{\mathrm{def}}(\{1\})$ admits an obvious splitting. Hence we conclude:

Corollary 5.1.2 *For any finitely generated discrete groups G and H, the diagram* *of homotopy groups*

$$\begin{tikzcd}
K^*_{\mathrm{def}}(G * H) \arrow[r] \arrow[d] & K^*_{\mathrm{def}}(G) \arrow[d] \\
K^*_{\mathrm{def}}(H) \arrow[r] & K^*_{\mathrm{def}}(\{1\})
\end{tikzcd}$$

is cartesian. For $*$ odd, $K^*_{\mathrm{def}}(\{1\}) = \pi_* \mathbf{ku} = 0$ and hence $K^*_{\mathrm{def}}(G * H) = K^*_{\mathrm{def}}(G) \oplus K^*_{\mathrm{def}}(H)$.

The proof of Theorem 5.1.1 requires a technical result regarding the topology of homotopy orbit spaces.

Lemma 5.1.3 *For any discrete groups G, H and K, the homotopy orbit space $\mathrm{Hom}(G *_K H, U(n))_{hU(n)}$ is naturally homeomorphic to the pullback*

$$\mathrm{Hom}(G, U(n))_{hU(n)} \times_{\mathrm{Hom}(K,U(n))_{hU(n)}} \mathrm{Hom}(H, U(n))_{hU(n)}.$$

The analogous statement holds for the general linear groups in place of the unitary groups.

Proof. It follows from the proof of Proposition 2.0.4 that for any group L, the space $\mathrm{Hom}(L, U(n))_{hU(n)}$ is homeomorphic to the realization of a simplicial space of the form

$$\left| k \mapsto U(n)^k \times \mathrm{Hom}(L, U(n)) \right|.$$

The lemma now follows from the fact that geometric realization commutes with pullbacks in the category of compactly generated spaces. The proof for $GL_n(\mathbb{C})$ is identical. □

Remark 5.1.4 *The above result actually holds much more generally, for homotopy orbit spaces formed using any reasonable model for the universal bundle. One cannot in general use simplicial spaces, though, and hence a somewhat lengthy point-set argument is required.*

Proof of Theorem 5.1.1. The proofs for the general linear and unitary cases are identical, so we work in the unitary case. The proof involves reducing to a diagram of homotopy orbit spaces, which will be homotopy cartesian by Lemma 5.1.3.

We begin by noting that since the spectra involved are all connective Ω-spectra, it will suffice to show that the diagram of zeroth spaces is homotopy cartesian. In order to apply Theorem 3.0.11 we need to filter the underlying monoids by submonoids which are stably group-like with respect to compatible representations. For each $n = 1, 2, \ldots$, we will define submonoids $\mathrm{Rep}(G * H)_{hU}^{(n)} \subset \mathrm{Rep}(G * H)_{hU}$, $\mathrm{Rep}(G)_{hU}^{(n)} \subset \mathrm{Rep}(G)_{hU}$ and $\mathrm{Rep}(H)_{hU}^{(n)} \subset \mathrm{Rep}(H)_{hU}$ having the following properties:

1. Each of these submonoids is a union of connected components in the larger monoid, and $\mathrm{Rep}(\cdot)_{hU} = \bigcup_n \mathrm{Rep}(\cdot)_{hU}^{(n)}$.

2. Under the natural maps from $\mathrm{Rep}(G * H)_{hU}$ to $\mathrm{Rep}(G)_{hU}$ and $\mathrm{Rep}(H)_{hU}$, $\mathrm{Rep}(G * H)_{hU}^{(n)}$ maps to $\mathrm{Rep}(G)_{hU}^{(n)}$ and to $\mathrm{Rep}(H)_{hU}^{(n)}$ (respectively).

3. There are representations ρ_n of G and ψ_n of H (of the same dimension $d = d(n)$) such that $\mathrm{Rep}(G*H)_{hU}^{(n)}$ is stably group-like with respect to $[*_d, (\rho_n, \psi_n)]$ and $\mathrm{Rep}(G)_{hU}^{(n)}$ and $\mathrm{Rep}(H)_{hU}^{(n)}$ are stably group-like with respect to $[*_d, \rho_n]$ and $[*_d, \psi_n]$ (respectively).

4. For each n, the square

$$\begin{tikzcd}
\mathrm{Rep}(G * H)_{hU}^{(n)} \arrow[r] \arrow[d] & \mathrm{Rep}(G)_{hU}^{(n)} \arrow[d] \\
\mathrm{Rep}(H)_{hU}^{(n)} \arrow[r] & \mathrm{Rep}(\{1\})_{hU}
\end{tikzcd}$$

is cartesian, i.e. the natural map

$$\mathrm{Rep}(G * H)_{hU}^{(n)} \longrightarrow \lim \left(\mathrm{Rep}(G)_{hU}^{(n)} \to \mathrm{Rep}(\{1\})_{hU} \leftarrow \mathrm{Rep}(H)_{hU}^{(n)} \right)$$

is a homeomorphism.

Assuming the existence of such filtrations, we now complete the proof of Theorem 5.1.1. By Lemma 2.0.3, it suffices to show that the diagram

$$\begin{tikzcd} \Omega B(\mathrm{Rep}(G*H)_{hU}) \arrow[r] \arrow[d] & \Omega B(\mathrm{Rep}(G)_{hU}) \arrow[d] \\ \Omega B(\mathrm{Rep}(H)_{hU}) \arrow[r] & \Omega B(\mathrm{Rep}(\{1\})_{hU}) \end{tikzcd} \tag{5.4}$$

is homotopy cartesian. It is easily seen, using properties (1) and (2) of the filtrations, that Diagram (5.4) is the colimit of the diagrams

$$\begin{tikzcd} \Omega B(\mathrm{Rep}(G*H)^{(n)}_{hU}) \arrow[r] \arrow[d] & \Omega B(\mathrm{Rep}(G)^{(n)}_{hU}) \arrow[d] \\ \Omega B(\mathrm{Rep}(H)^{(n)}_{hU}) \arrow[r] & \Omega B(\mathrm{Rep}(\{1\})_{hU}) \end{tikzcd} \tag{5.5}$$

(as n tends to infinity). Hence it will suffice to show that for each n, Diagram (5.5) is homotopy cartesian.

Now, by property (3) we know that there are representations $\rho_n\colon G \to U(d(n))$ and $\psi_n\colon H \to U(d(n))$ such that these monoids are stably group-like with respect to the points $[*_{d(n)}, (\rho_n, \psi_n)]$, $[*_{d(n)}, \rho_n]$ and $[*_{d(n)}, \psi_n]$ (respectively). Furthermore, the proof of Corollary 3.0.16 shows that these basepoints are anchored, so we may apply Theorem 3.0.11. To simplify notation, we let $X^{(n)} = \coprod_{k=0}^{\infty} BU(k)$, $Y^{(n)} = \mathrm{Rep}(G)^{(n)}_{hU}$, $Z^{(n)} = \mathrm{Rep}(H)^{(n)}_{hU}$, and $W^{(n)} = \mathrm{Rep}(G*H)^{(n)}_{hU}$. Also, let

$$W^{(n)}_{\infty} = \mathrm{colim}\left(W^{(n)} \xrightarrow{\oplus[*_{d(n)},(\rho_n,\psi_n)]} W^{(n)} \xrightarrow{\oplus[*_{d(n)},(\rho_n,\psi_n)]} \cdots \right)$$

and let $\widetilde{W}^{(n)}_{\infty}$ denote the homotopy colimit of the same sequence. We define $X^{(n)}_{\infty}$, $\widetilde{X}^{(n)}_{\infty}$, $Y^{(n)}_{\infty}$, $\widetilde{Y}^{(n)}_{\infty}$, $Z^{(n)}_{\infty}$, and $\widetilde{Z}^{(n)}_{\infty}$ analogously; the direct system for X uses block sum with $*_{d(n)} \in BU(d(n))$.

With this notation, Theorem 3.0.11 shows that for any n, Diagram (5.5) is homotopy cartesian if and only if the diagram

$$\begin{tikzcd} \widetilde{W}_\infty^{(n)} \arrow[r] \arrow[d] & \widetilde{Y}_\infty^{(n)} \arrow[d] \\ \widetilde{Z}_\infty^{(n)} \arrow[r] & \widetilde{X}_\infty^{(n)} \end{tikzcd} \tag{5.6}$$

is homotopy cartesian. Note that here the naturality statement in Theorem 3.0.11 is extremely important. The fact that Diagram (5.6) is homotopy cartesian essentially follows from property (4) of the filtrations together with the general fact that homotopy pull-backs commute with directed homotopy colimits. In this case, though, we can provide the following direct argument.

We must show that the natural map

$$\widetilde{W}_\infty^{(n)} \longrightarrow \operatorname{holim}\left(\widetilde{Y}_\infty^{(n)} \longrightarrow \widetilde{X}_\infty^{(n)} \longleftarrow \widetilde{Z}_\infty^{(n)}\right)$$

is a weak equivalence. But this map fits into the commutative diagram

$$\begin{tikzcd} \widetilde{W}_\infty^{(n)} \arrow[rr] \arrow[d, "\wr"] & & \operatorname{holim}\left(\widetilde{Y}_\infty^{(n)} \longrightarrow \widetilde{X}_\infty^{(n)} \longleftarrow \widetilde{Z}_\infty^{(n)}\right) \arrow[dd, "\wr"] \\ W_\infty^{(n)} \arrow[d, "\cong"] & & \\ \lim\left(Y_\infty^{(n)} \longrightarrow X_\infty^{(n)} \longleftarrow Z_\infty^{(n)}\right) \arrow[rr, "\alpha"] & & \operatorname{holim}\left(Y_\infty^{(n)} \longrightarrow X_\infty^{(n)} \longleftarrow Z_\infty^{(n)}\right). \end{tikzcd} \tag{5.7}$$

The maps labeled $\simeq$ are clearly weak equivalences, since they arise from collapsing mapping telescopes. Property (4) states that $W^{(n)} \cong \lim(Y^{(n)} \to X^{(n)} \leftarrow Z^{(n)})$, and hence (after unwinding the notation) one sees that the homeomorphism on the right comes from interchanging a colimit and a limit. To see that the bottom map α is a weak equivalence, note that the maps $Z_n \to X_n$ are Serre fibrations (in each component, this map is just the map from a homotopy orbit space $C_{hU(k)}$ to $BU(k)$), and a colimit of Serre fibrations is a Serre fibration. Hence $Z_\infty^{(n)} \to X_\infty^{(n)}$ (and similarly $Y_\infty^{(n)} \to X_\infty^{(n)}$) is a Serre fibration. It is a well-known fact that if $f\colon E \to B$ is a Serre fibration, then for any map $g\colon A \to B$ there is a weak equivalence

$$\lim\left(A \xrightarrow{g} B \xleftarrow{f} E\right) \xrightarrow{\simeq} \operatorname{holim}\left(A \xrightarrow{g} B \xleftarrow{f} E\right),$$

and this precisely tells us that α is a weak equivalence. Since all of the other maps in Diagram (5.7) are weak equivalences, so is the top map.

To complete the proof, we must construct filtrations satisfying the four properties listed above. Given a topological monoid M and a submonoid $A \subset \pi_0(M)$, we have a corresponding submonoid $M(A) \subset M$ generated by all elements representing components in A. Clearly $\pi_0(M(A)) = A$. Now, let $C_n(G) \subset \pi_0\mathrm{Rep}(G)_{hU} = \pi_0\mathrm{Rep}(G)$ denote the submonoid generated by all representations of dimension at most n. We set $\mathrm{Rep}(G)_{hU}^{(n)} = \mathrm{Rep}(G)_{hU}(C_n(G))$, and we define $C_n(H)$ and $\mathrm{Rep}(H)_{hU}^{(n)}$ similarly.

The unitary and general linear representation spaces of any finitely generated group are real algebraic varieties, cut out by the ideal corresponding to the group relations. Hence these spaces are triangulable [21], which implies that their path components and their connected components coincide. In the unitary case, the representation spaces $\mathrm{Hom}(G, U(m))$ and $\mathrm{Hom}(H, U(m))$ are always compact, hence have finitely many (path) components. This implies that $C_n(G)$ and $C_n(H)$ are finitely generated in the unitary case. More generally, Whitney's theorem [49] states that any (real) algebraic variety has finitely many connected components, and hence these monoids are finitely generated in the general linear case as well.

Now, choose generators $[\rho_1], \ldots, [\rho_{r_n}]$ for $C_n(G)$ and $[\psi_1], \ldots, [\psi_{q_n}]$ for $C_n(H)$. Of course, we may assume that all of these representations have dimension at most n. We define $C_n(G * H) \subset \pi_0(\mathrm{Rep}(G * H))$ to be the pullback of $C_n(G)$ and $C_n(H)$ over $\mathbb{N} = \pi_0\mathrm{Rep}(\{1\})_{hU}$, i.e. the submonoid generated by all components of the form

$$[\bigoplus_{i=1}^{r_n} \rho_i^{a_i}, \bigoplus_{j=1}^{q_n} \psi_j^{b_j}].$$

We set $\mathrm{Rep}(G * H)_{hU}^{(n)} = \mathrm{Rep}(G * H)_{hU}\left(C_n(G * H)\right)$. It is easily checked that the components of this monoid are in fact generated by the finite subset

$$F_n = \{[\bigoplus_i \rho_i^{a_i}, \bigoplus_j \psi_j^{b_j}] : \text{ either } a_i < n\, \forall i, \text{ or } b_j < n\, \forall j\}.$$

(When $n = 1$, we must replace $<$ by $\leqslant$.)

Properties (1) and (2) are now immediate from the definitions. To prove property (3), note that $\mathrm{Rep}(G * H)^{(n)}_{hU}$ is automatically stably group-like with respect to the sum of (the obvious representatives for) the elements in the generating set F_n. Letting $[\alpha_n, \beta_n]$ denote this sum, we need to check that $\mathrm{Rep}(G)^{(n)}_{hU}$ is stably group-like with respect to α_n and similarly for H. But this follows immediately from the fact the every ρ_i $(i = 1, \ldots, r_n)$ appears as a summand in α_n (note that some ψ_j must be one-dimensional), and similarly for the ψ_j. Finally, property (4) follows easily from the definitions, using Lemma 5.1.3. This completes the proof of Theorem 5.1.1. □

As an application of Theorem 5.1.1, we now compute $K^*_{\mathrm{def}}(PSL_2(\mathbb{Z}))$ in the unitary case. It is well known that $PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$ (see [5] for a short proof). We work with $PSL_2(\mathbb{Z})$ for concreteness, although the same argument gives a computation of $K^*_{\mathrm{def}}(G * H)$ for any finite groups G and H.

Lawson [29] has shown, using basic representation theory, that for any finite group G, $K_{\mathrm{def}}(G) \simeq \bigvee_k \mathbf{ku}$, where k is the number of irreducible representations of G. Hence in particular

$$K^*_{\mathrm{def}}(\mathbb{Z}/m) = \begin{cases} 0, & * \text{ odd}, \\ \mathbb{Z}^m, & * \text{ even}. \end{cases}$$

By Remark 5.1.2, we now have (for any $i \geqslant 0$) $K^{2i+1}_{\mathrm{def}}(\mathbb{Z}/2 * \mathbb{Z}/3) \cong 0 \oplus 0 = 0$ and an exact sequence

$$0 \longrightarrow K^{2i}_{\mathrm{def}}(\mathbb{Z}/2 * \mathbb{Z}/3) \longrightarrow \mathbb{Z}^2 \oplus \mathbb{Z}^3 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

from which it follows that $K^{2i}_{\mathrm{def}}(\mathbb{Z}/2 * \mathbb{Z}/3) \cong \mathbb{Z}^4$. Thus we have:

Proposition 5.1.5

$$K^*_{\mathrm{def}}(PSL_2(\mathbb{Z})) = \begin{cases} 0, & * \text{ odd}, \\ \mathbb{Z}^4, & * \text{ even}. \end{cases}$$

We briefly indicate Lawson's computation of $K_{\text{def}}(G)$ for G finite. Any representation ρ of G breaks up canonically into isotypical components, and together with Schur's Lemma this gives a permutative functor $\mathcal{R}(G) \to \text{Vect}^k$, which records the dimensions of the isotypical components. Here k is the number of irreducible representations of G and Vect is the category with $\mathbb{N}$ as objects and $\coprod_n U(n)$ as morphisms (we will work in the unitary case, but the general linear case is identical). This functor is continuous, since any two representations connected by a path are isomorphic (since G is finite, the trace gives a continuous, complete invariant of the isomorphism type, and it can take on only countably many values). One now checks that this functor induces a weak equivalence on classifying spaces, and hence on K–theory spectra. This is rather like the proof of Proposition 2.0.4: one sees that $B\mathcal{R}(G)$ is a model for $\coprod_{\rho_i} B\left(\text{Stab}(\rho_i)\right)$, where the ρ_i are representatives for the isomorphism types. Now Schur's Lemma implies that $\text{Stab}(\rho_i) \cong \prod U(n_j)$, where the n_j are the dimensions of the isotypical components of ρ_i. The comparison with $B(\text{Vect}^k) \cong B(\text{Vect})^k \cong \left(\coprod_n BU(n)\right)^k$ is now straightforward.

5.2 Reduction of excision to representation varieties, and an example

The first goal of this section is to reduce the question of excision to representation varieties, at least when the groups in question have stably group-like representation monoids (in an appropriately compatible manner). In other words, we wish to show that information about the maps

$$\text{Hom}(G *_K H, U(n)) \xrightarrow{\phi} \text{holim} \begin{pmatrix} \text{Hom}(H, U(n)) \\ \downarrow \\ \text{Hom}(G, U(n)) \\ \uparrow \\ \text{Hom}(K, U(n)) \end{pmatrix}$$

allows us to deduce information about the map

$$K_{\mathrm{def}}(G *_K H) \xrightarrow{\Phi} \operatorname{holim} \begin{pmatrix} K_{\mathrm{def}}(H) \\ \downarrow \\ K_{\mathrm{def}}(G) \\ \uparrow \\ K_{\mathrm{def}}(K) \end{pmatrix}.$$

Before stating the result precisely, we discuss some preliminary lemmas.

We will be comparing the long exact sequences associated to certain fibrations, using the 5-lemma. In low degrees, some caution is necessary, since π_1 is not in general an abelian group. For convenience of the reader, we record the necessary result.

Lemma 5.2.1 *Consider a commutative diagram of (possibly non-abelian) groups*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, "d"] & E \arrow[d, "e"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

in which the rows are exact.

If a is surjective and both b and d are injective, then c is injective.

If b and d are surjective and e is injective, then c is surjective.

In order to pass from representation varieties to deformation K-theory, we first need to deduce results about homotopy orbit spaces. This will be done by studying the fibration $X \to X_{hG} \to BG$ associated to a G-space X. We need a simple fact about fibrations and homotopy limits. (A more general statement is possible, with essentially the same proof.) In order to state the result, we make the following definition:

Definition 5.2.2 *We call a map $f : X \to Y$ of based spaces (l, k)-connected ($0 \leqslant l \leqslant k$) if $f_* : \pi_n X \to \pi_n Y$ is an isomorphism for $l \leqslant n \leqslant k$, a surjection for*

$n = k+1$, and an injection for $n = l-1$. We call a commutative square of spaces

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

(l,k)-cartesian if the natural map

$$X \longrightarrow \operatorname{holim}(Y \longrightarrow Z \longleftarrow W)$$

is (l,k)-connected.

We allow $k = \infty$ and $l = 0$, and we set $\pi_{-1}Z = 0$ for any space Z, so that the injectivity condition is vacuous and hence $(0,k)$-connectivity is the standard notion of k-connectivity. The above definition is useful since we are interested in comparing $K^*_{\text{def}}(G)$ with $K^*(BG)$, and in certain cases (e.g. fundamental groups of Riemann surfaces) these groups are known to be isomorphic only above a certain dimension (hence we don't want to consider only ordinary connectivity of maps).

Lemma 5.2.3 *Let $\mathcal{G}$ be a connected group. Then a commutative square of $\mathcal{G}$-spaces (with all maps equivariant)*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

is (l,k)-cartesian if and only if the diagram of homotopy orbit spaces

$$\begin{tikzcd} X_{h\mathcal{G}} \arrow[r] \arrow[d] & Y_{h\mathcal{G}} \arrow[d] \\ Z_{h\mathcal{G}} \arrow[r] & W_{h\mathcal{G}} \end{tikzcd}$$

is (l,k)-cartesian.

Proof. Consider the commutative diagram of fibrations

$$\begin{tikzcd}[row sep=small, column sep=small]
& Y \arrow[rr] \arrow[dd] & & Y_{h\mathcal{G}} \arrow[rr] \arrow[dd] & & B\mathcal{G} \arrow[dd] \\
X \arrow[ur] \arrow[rr] \arrow[dd] & & X_{h\mathcal{G}} \arrow[ur] \arrow[rr] \arrow[dd] & & B\mathcal{G} \arrow[ur] \arrow[dd] & \\
& W \arrow[rr] & & W_{h\mathcal{G}} \arrow[rr] & & B\mathcal{G}, \\
Z \arrow[ur] \arrow[rr] & & Z_{h\mathcal{G}} \arrow[ur] \arrow[rr] & & B\mathcal{G} \arrow[ur] &
\end{tikzcd}$$

in which all the maps $B\mathcal{G} \to B\mathcal{G}$ are the identity. Let $\widetilde{B\mathcal{G}}$ denote the homotopy limit holim $\left(B\mathcal{G} \overset{=}{\to} B\mathcal{G} \overset{=}{\leftarrow} B\mathcal{G}\right)$, and note that there is an obvious homotopy equivalence $B\mathcal{G} \overset{\simeq}{\to} \widetilde{B\mathcal{G}}$. Let

$$X \xrightarrow{\phi} \operatorname{holim}(Y \to W \leftarrow Z) \text{ and } X_{h\mathcal{G}} \xrightarrow{\Phi} \operatorname{holim}(Y_{h\mathcal{G}} \to W_{h\mathcal{G}} \leftarrow Z_{h\mathcal{G}})$$

denote the natural maps. Consider the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\phi"] & X_{h\mathcal{G}} \arrow[r] \arrow[d, "\Phi"] & B\mathcal{G} \arrow[d, "\simeq"] \\
\operatorname{holim}(Y \to W \leftarrow Z) \arrow[r] \arrow[d, dotted, "\iota"] & \operatorname{holim}(Y_{h\mathcal{G}} \to W_{h\mathcal{G}} \leftarrow Z_{h\mathcal{G}}) \arrow[r, "\alpha"] \arrow[d, "\simeq"] & \widetilde{B\mathcal{G}}, \\
\operatorname{hofib}(\alpha) \arrow[r] & P_\alpha \arrow[ur] &
\end{tikzcd}$$

in which P_α denotes the total space of the fibration associated to α, and the map ι exists because the composite along the middle row is constant. We claim that the map ι is a weak equivalence, i.e. that the middle row is a homotopy fibration. Assuming this, the lemma follows easily by applying Lemma 5.2.1 to the resulting diagram of long exact sequences in homotopy. (Note that since we have assumed $\mathcal{G}$ is connected, $\pi_1 B\mathcal{G} = 0$. Hence these long exact sequences can be cut off after the π_1 stage, and we need not worry about applying the five lemma to a diagram containing sets. Moreover, π_0 is easily dealt with since $\pi_1 B\mathcal{G} = \pi_0 B\mathcal{G} = 0$ implies that the maps on π_0 induced by ϕ and Φ are isomorphic.)

To see that ι is a weak equivalence, note for any $\mathcal{G}$-space T, the natural inclusion $T \hookrightarrow \operatorname{hofib}(T_{h\mathcal{G}} \to B\mathcal{G})$ is (obviously) a weak homotopy equivalence, and hence the induced map

$$\operatorname{holim}\begin{pmatrix} Y \\ \downarrow \\ W \\ \uparrow \\ Z \end{pmatrix} \xrightarrow{\Psi} \operatorname{holim}\begin{pmatrix} \operatorname{hofib}(Y_{h\mathcal{G}} \to B\mathcal{G}) \\ \downarrow \\ \operatorname{hofib}(W_{h\mathcal{G}} \to B\mathcal{G}) \\ \uparrow \\ \operatorname{hofib}(Z_{h\mathcal{G}} \to B\mathcal{G}) \end{pmatrix}.$$

is a homotopy equivalence as well. Now ι is simply the composition of Ψ with the natural homeomorphism

$$\operatorname{holim}\begin{pmatrix} \operatorname{hofib}(Y_{h\mathcal{G}} \to B\mathcal{G}) \\ \downarrow \\ \operatorname{hofib}(W_{h\mathcal{G}} \to B\mathcal{G}) \\ \uparrow \\ \operatorname{hofib}(Z_{h\mathcal{G}} \to B\mathcal{G}) \end{pmatrix} \xrightarrow{\cong} \operatorname{hofib}\left(\operatorname{holim}\begin{pmatrix} Y_{h\mathcal{G}} \\ \downarrow \\ W_{h\mathcal{G}} \\ \uparrow \\ Z_{h\mathcal{G}} \end{pmatrix} \longrightarrow \operatorname{holim}\begin{pmatrix} B\mathcal{G} \\ \downarrow \| \\ B\mathcal{G} \\ \uparrow \| \\ B\mathcal{G} \end{pmatrix}\right),$$

so ι is a homotopy equivalence as well. $\square$

We are now ready to discuss our reduction of the excision problem to representation varieties. Given an amalgamation diagram

$$\begin{tikzcd} K \arrow[r, "f_1"] \arrow[d, "f_2"] & G \arrow[d, "h"] \\ H \arrow[r, "k"] & G *_K H, \end{tikzcd}$$

we say that $\operatorname{Rep}(G *_K H)$ is compatibly stably group-like if the representation monoids $\operatorname{Rep}(-)$ are stably group-like with respect to representations which map to one another via the natural restriction maps.

Proposition 5.2.4 *Assume that* $\operatorname{Rep}(G *_K H)$ *is compatibly stably-grouplike. If* *the natural map*

$$\mathrm{Hom}(G *_K H, U(n)) \xrightarrow{\phi} \operatorname{holim} \begin{pmatrix} \mathrm{Hom}(H, U(n)) \\ \downarrow \\ \mathrm{Hom}(G, U(n)) \\ \uparrow \\ \mathrm{Hom}(K, U(n)) \end{pmatrix}$$

is (l,k)-connected for infinitely many n, then the natural map

$$K_{\mathrm{def}}(G *_K H) \xrightarrow{\Phi} \operatorname{holim} \begin{pmatrix} K_{\mathrm{def}}(H) \\ \downarrow \\ K_{\mathrm{def}}(G) \\ \uparrow \\ K_{\mathrm{def}}(K) \end{pmatrix}$$

is (l,k)-connected as well.

Proof. Theorem 3.0.11 allows us to replace the diagram of deformation K-theory spectra with a diagram of infinite mapping telescopes formed from the representation monoids $\mathrm{Rep}(-)_{hU}$ (via block sum with the appropriate representations). Since we are only interested in the components containing the basepoints, and since the homotopy groups of a telescope are simply the colimit of the homotopy groups of the spaces involved, it suffices to show that the diagrams

$$\begin{array}{ccc} \mathrm{Hom}(G *_K H, U(n))_{hU(n)} & \longrightarrow & \mathrm{Hom}(H, U(n))_{hU(n)} \\ \downarrow & & \downarrow \\ \mathrm{Hom}(H, U(n))_{hU(n)} & \longrightarrow & \mathrm{Hom}(K, U(n)) \end{array}$$

are (l,k) cartesian for infinitely many n. But by Lemma 5.2.3, this is equivalent to the hypothesis that ϕ is (l,k)-connected for infinitely many n. □

We now consider a simple example to which Proposition 5.2.4 applies. Let Γ_k

be the group

$$\Gamma_k = \{a, b | a^k = b^k\}.$$

Then we have the following description of Γ_k as an amalgamated product:

$$\Gamma_k \cong \operatorname{colim}(\mathbb{Z} \xleftarrow{\mu_k} \mathbb{Z} \xrightarrow{\mu_k} \mathbb{Z}),$$

where $\mu_k : \mathbb{Z} \to \mathbb{Z}$ denotes multiplication by k. Note that Γ_2 is the fundamental group of $\mathbb{R}P^2 \# \mathbb{R}P^2$, which is homeomorphic to the Klein bottle. We will prove later (Proposition 6.1.11) that $\operatorname{Rep}(\Gamma_k)$ is stably group-like (with respect to the trivial representation $1 \in \operatorname{Hom}(\Gamma_k, U(1))$).

Proposition 5.2.5 *The natural map*

$$\Phi : K_{\mathrm{def}}(\Gamma_k) \to \operatorname{holim}\left(K_{\mathrm{def}}(\mathbb{Z}) \xrightarrow{\mu_k^*} K_{\mathrm{def}}(\mathbb{Z}) \xleftarrow{\mu_k^*} K_{\mathrm{def}}(\mathbb{Z})\right)$$

is 0*-connected, i.e.* Φ *induces an isomorphism on* π_0 *and a surjection on* π_1*.*

We make the following conjecture.

Conjecture 5.2.6 *For any* $k \in \mathbb{N}$ *the group* Γ_k *satisfies excision for deformation* K*-theory, i.e. the map* Φ *in Proposition 5.2.5 is a weak equivalence.*

We note that for $k = 2$, the group Γ_2 is isomorphic to the fundamental group of $\mathbb{R}P^2 \# \mathbb{R}P^2$. In this case, it follows from Theorem 4.4.1 that the domain and range of Φ have abstractly isomorphic homotopy groups (and Conjectures 5.2.6 and 5.0.7 coincide).

The following proof is a bit ad-hoc. A more natural approach, based on transversality and stratified systems of fibrations, is described afterwards.

Proof of Proposition 5.2.5. Proposition 5.2.4 reduces the problem to the study of the representation varieties $\operatorname{Hom}(\Gamma_k, U(n))$, since by Proposition 6.1.11, $\operatorname{Rep}(\Gamma_k)$ is stably group-like. (It is immediate that $\operatorname{Rep}(\mathbb{Z})$ is stably grouplike.)

For ease of notation, let us define

$$\mathcal{H}(k, n) = \operatorname{holim}(U(n) \xrightarrow{p_k} U(n) \xleftarrow{p_k} U(n)),$$

where p_k denotes the k^{th} power map $A \mapsto A^k$ (this is, of course, the map on representations induced by μ_k). We just need to show that the natural map

$$\Psi : \mathrm{Hom}(\Gamma_k, U(n)) \to \mathcal{H}(k, n)$$

induces an isomorphism on π_0 and a surjection on π_1.

First we consider π_0. From the long-exact Mayer-Vietoris sequence associated to the homotopy pullback $\mathcal{H}(k, n)$, one immediately sees that $\mathcal{H}(k, n)$ has precisely k connected components. By Proposition 6.1.11, we know that $\mathrm{Hom}(\Gamma_k, U(n))$ also has k components, and hence it will suffice to show that the natural map

$$\Psi : \mathrm{Hom}(\Gamma_k, U(n)) \to \mathcal{H}(k, n)$$

is surjective on π_0. Points in $\mathcal{H}(k, n)$ are described by triples (A, B, γ), with $A, B \in U(n)$ and γ a path in $U(n)$ from A^k to B^k. Using the fact that $U(n)$ is connected, one may easily construct a path in $\mathcal{H}(k, n)$ connecting any such point (A, B, γ) to a point of the form (I_n, I_n, β). Now, β is a loop in $U(n)$ based at the identity, and hence is homotopic to a loop of the form $\alpha^m(t) = e^{2\pi imt} \oplus I_{n-1}$ for some $m \in Z$. Hence each path component of $\mathcal{H}(k, n)$ contains a point of the form (I, I, α^m) for some $m \in \mathbb{Z}$. We will show that each of these points is connected by a path to a representation, i.e. a point of the form (A, B, c_{A^k}) where c_{A^k} is the constant path at $A^k = B^k$.

Let $\tilde{\alpha}^m(t) = e^{2\pi im(\frac{k-1}{k} + \frac{t}{k})} \oplus I_{n-1}$. Let α^m_r denote the loop α^m restricted to the interval $[0, r]$ (and reparametrized on the interval $[0, 1]$). We define a path in $\mathcal{H}(k, n)$ by the formula

$$s \mapsto (I_n, \tilde{\alpha}^m(1 - s), \alpha^m_{1-s}).$$

One easily checks that this defines a path in $\mathcal{H}(k, n)$, starting at (I_n, I_n, α^m) and ending at $(I_n, e^{2\pi i\frac{k-1}{k}m} \oplus I_{n-1}, c_{I_n})$. This completes the proof that Ψ induces an isomorphism on π_0.

The proof of surjectivity on π_1 is a simple algebraic calculation. In fact, we will show that the map

$$\Psi_* : \pi_{2l+1}\mathrm{Hom}(\Gamma_k, U(n)) \to \pi_{2l+1}\mathcal{H}(k, n)$$

is surjective for any $k \geq 2$ and any $l < n$.

Since $\mathrm{Hom}(\Gamma_k, U(n))$ is the limit of the diagram describing $\mathcal{H}(k, n)$, we have a commutative diagram

$$\begin{tikzcd}
\pi_{2l+2}U(n) \arrow[r] & \pi_{2l+1}\mathcal{H}(k,n) \arrow[r] & \pi_{2l+1}U(n) \oplus \pi_{2l+1}U(n) \arrow[r] & \pi_{2l+1}U(n) \\
& \pi_{2l+1}\mathrm{Hom}(\Gamma_k, U(n)) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

in which the top line comes from the Mayer-Vietoris sequence for the homotopy pullback $\mathcal{H}(k, n)$. Since $\pi_{2l+2}U(n) = 0$, this diagram gives us a commutative diagram

$$\begin{tikzcd}
\pi_{2l+1}\mathcal{H}(k,n) \arrow[r, "\cong"] & \ker\Big(\pi_{2l+1}U(n) \oplus \pi_{2l+1}U(n) \longrightarrow \pi_{2l+1}U(n)\Big) \\
\pi_{2l+1}\mathrm{Hom}(\Gamma_k, U(n). \arrow[u] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

So it suffices to check that the map ψ is surjective. But the map $\pi_{2l+1}U(n) \oplus \pi_{2l+1}U(n) \longrightarrow \pi_{2l+1}U(n)$ is simply $(\alpha, \beta) \mapsto (p_k)_*\alpha - (p_k)_*\beta$, and since p_k is the k^{th} power map on the group $U(n)$, $(p_k)_*$ is the k^{th} power map on the homotopy group $\pi_{2l+1}U(n) \cong \mathbb{Z}$. Hence the kernel is precisely the diagonal subgroup of $\pi_{2l+1}U(n) \oplus \pi_{2l+1}U(n)$. If we choose a generator α for $\pi_{2l+1}U(n)$, then the image of the map $(\alpha, \alpha) : S^{2l+1} \to U(n) \times U(n)$ lies inside $\mathrm{Hom}(\Gamma_k, U(n))$, and clearly $\psi(\alpha, \alpha) = (\alpha, \alpha)$. This shows that ψ is surjective, and hence completes the proof of Proposition 5.2.5. $\square$

We now describe a less ad-hoc approach to the excision problem for the groups Γ_k. Unfortunately, this approach does not immediately give any further information, and hence we omit the proofs. We hope, however, that in other situations this technique might be useful.

In order to simplify notation, we will work with $k = 2$ from now on. The central idea is that the squaring map $p_2 : U(n) \to U(n)$ can be given the structure of a stratified system of fibrations, in the sense of Quinn [39], and one may then attempt to mimic the simplest possible proof that any pull-back square in which one map is a fibration is homotopy-cartesian.

More precisely, we have a stratification of $U(n)$ by disjoint submanifolds, over which the squaring map is a fiber-bundle whose fibers are manifolds. Moreover, the strata (and their "square roots") have neighborhoods homeomorphic to mapping cylinders, in a manner compatible with μ_2. One then wants to use the lifting properties of the various strata to analyze homotopy groups. The difficulty comes in interpolating between the lifting properties of the various fibrations appearing in our stratified system. One might hope that this could be handled by careful use of the mapping cylinder neighborhoods of the strata (which themselves have convenient lifting properties). Unfortunately we have no concrete results along these lines at the moment.

We begin by describing the *eigenvalue stratification* of $U(n)$, which seems to be well-known. Before stating the result precisely, we introduce some terminology. Let $A \in U(n)$ be any unitary matrix. We define the *eigenpartition* of A, denoted $e(A)$, to be the partition of n whose components are the dimensions of the eigenspaces of A. By convention, we will always represent partitions as vectors $\vec{\mathbf{p}} = (p_1, \ldots, p_m)$ with $1 \leq p_1 \leq \cdots \leq p_m$, and we denote the number of terms in $\vec{\mathbf{p}}$ by $l(p)$.

For any partition $\vec{\mathbf{p}}$ we define

$$E_{\vec{\mathbf{p}}} = \{A \in U(n) \,|\, e(A) = \vec{\mathbf{p}}\}.$$

In order to describe these subspaces, we need some further terminology.

Definition 5.2.7 *Given a vector $\vec{\mathbf{v}} = (v_1, \ldots, v_m) \in \mathbb{Z}^m$, we define a subgroup $\Sigma_{\vec{\mathbf{v}}}$ of the symmetric group Σ_m on m letters by setting*

$$\Sigma_{\vec{\mathbf{v}}} = \{\sigma \in \Sigma_m \,|\, v_{\sigma(i)} = v_i, \ i = 1, \ldots m\}.$$

For any partition $\vec{\mathbf{p}}$ of n, we denote the manifold of orthogonal flags in $\mathbb{C}^n$ with *dimensions* p_i *by* $\mathrm{Flag}(\vec{\mathbf{p}})$ *(so* $\mathrm{Flag}(\vec{\mathbf{p}}) \cong U(n)/\prod_i U(p_i)$*). Note that the group* $\Sigma_{\vec{\mathbf{p}}}$ *acts naturally on* $\mathrm{Flag}(\vec{\mathbf{p}})$ *by permuting subspaces of equal dimension.*

We now have:

Proposition 5.2.8 *For any partition* $\vec{\mathbf{p}}$ *of* n *with* $l(\vec{\mathbf{p}}) = l$, $E_{\vec{\mathbf{p}}}$ *is a submanifold of* $U(n)$ *(neither open nor closed, in general) and*

$$E_p \cong \left((S^1)^l - \widetilde{\Delta}) \times \mathrm{Flag}(\vec{\mathbf{p}})\right)/\Sigma_{\vec{\mathbf{p}}},$$

where

$$\widetilde{\Delta} = \{(\lambda_1, \ldots, \lambda_l) \in (S^1)^l \,|\, \lambda_i = \lambda_j \text{ for some } i \neq j\}$$

denotes the "fat diagonal," and $\Sigma_{\vec{\mathbf{p}}}$ *acts diagonally.*

We will only need a small part of the following proposition, but we state the full result anyhow.

Proposition 5.2.9 *The map* $p_k : p_k^{-1}(E_{\vec{\mathbf{p}}}) \to E_{\vec{\mathbf{p}}}$ *is a fiber bundle. In the case* $k = 2$*, the fiber is*

$$\left(\coprod_{\vec{\mathbf{i}} \leqslant \vec{\mathbf{p}}} \prod_{j=1}^{l(\vec{\mathbf{p}})} Gr_{i_j}(\mathbb{C}^{p_j})\right)^2,$$

where $\vec{\mathbf{i}} \leqslant \vec{\mathbf{p}}$ *simply means that these partitions have the same length and* $i_j \leqslant p_j$ *for each* j.

Remark 5.2.10 *It is not difficult to work out the fibers for* $k > 2$*, but the notation becomes cumbersome. The proof of this result is rather like the proof that the universal bundle over the Grassmanian is locally trivial.*

We now explain how to use these results to give another proof that the map

$$\pi_0 \mathrm{Hom}(\Gamma_k, U(n)) \longrightarrow \pi_0 \mathcal{H}(k, n)$$

is surjective. Points in $\mathcal{H}(k, n)$ may be represented by triples (A, B, γ) where $A, B \in U(n)$ and γ is a path in $U(n)$ from A^k to B^k. First, note that since $U(n)$ is connected we may assume that A^k and B^k lie in the open stratum $E_{(1,\ldots,1)}$. Next, note that if we homotope the path γ this also does not change the path component of our point. Hence we may assume that γ is smooth and transverse to each stratum $E_{\vec{\mathbf{p}}}$. The following easy computation of codimensions shows that this in fact allows us to assume γ lies in $E_{(1,\ldots,1)}$.

Lemma 5.2.11 *For any length k partition $\vec{\mathbf{p}}$ of n, the codimension of $E_{\vec{\mathbf{p}}}$ in $U(n)$ is $\sum_{i=1}^{k}(p_i^2 - 1)$, and hence this codimension is bounded below by $3 \cdot \#\{j : p_j > 1\}$.*

Now, by Proposition 5.2.9 we know that p_k restricts to a fiber bundle (in fact, a covering map) over the open stratum $E_{(1,\ldots,1)}$. Hence we may lift γ (or rather the reversed path $\overline{\gamma}$) to a path $\tilde{\gamma}$ starting at B and ending at some k^{th} root A' of A^k. This provides us with a path $t \mapsto (A, \tilde{\gamma}(t), \gamma|_{[0,1-t]})$ in $\mathcal{H}(k,n)$ from (A, B, γ) to (A, A', c_{A^k}) (where c_{A^k} denotes the constant path). Since the latter element is in the image of $\mathrm{Hom}(\Gamma_k, U(n))$, this completes our second proof of surjectivity on π_0.

Chapter 6

Examples

In this section we examine various cases in which the monoid underlying deformation K-theory is stably group-like with respect to some representation ρ. The first two results deal with groups modeled on the fundamental groups of (possibly non-orientable) compact surfaces, and these results deal exclusively with unitary deformation K-theory. In these first two cases, we show that the monoid $\mathrm{Rep}(G)$ is stably group-like with respect to the trivial representation $1 \in \mathrm{Hom}(G, U(1))$, so that Corollary 3.0.16 gives a particularly simple model for the zeroth space of $K_{\mathrm{def}}(G)$. The third result deals with finitely generated abelian groups, and here we find that the monoid underlying deformation K-theory (in both the unitary and the general linear case) is stably group-like with respect to a larger representation (the sum of the characters of the torsion subgroup). All three results yield computations of $K^0_{\mathrm{def}}(G)$.

We will say that $\mathrm{Rep}(G)$ is stably group-like if it is stably group-like with respect to the trivial representation $1 \in \mathrm{Hom}(G, U(1))$. Recall that this means that for every representation ρ there is a representation ψ (a stable homotopy inverse for ρ) such that $\rho \oplus \psi$ lies in the connected component of the trivial representation.

The first two types of group we will study are both modeled on fundamental groups of (possibly non-orientable) surfaces. In each case we will distinguish path components using a simple obstruction to the existence of paths between representations. This obstruction was originally defined for fundamental groups of non-orientable surfaces by Ho and Liu [24, 25]. In the second section, we discuss finitely generated abelian groups.

6.1 Surface-type groups

We begin with a general discussion of Ho and Liu's obstruction. First we note that there is a particularly simple model for the universal cover of the unitary group.

Lemma 6.1.1 *The homomorphism $p_n : \mathbb{R} \times SU(n) \to U(n)$ given by $(t, A) \mapsto e^{2\pi i t} A$ is a universal covering map for $U(n)$.*

Proof. The group $\mathbb{R} \times SU(n)$ is simply connected, and $\ker(p_n)$ is the discrete subgroup generated by $(1/n, e^{-\frac{2\pi i}{n}})$. □

From now on we will denote $\mathbb{R} \times SU(n)$ by $\widetilde{U(n)}$, and $p_n : \widetilde{U(n)} \to U(n)$ will be the above homomorphism. In addition, we will identify $\ker(p_n)$ with $\mathbb{Z}$ via the map

$$(1/n, e^{-\frac{2\pi i}{n}}) \mapsto 1.$$

In order to describe the structure of π_0 in our examples, we need some notation and a lemma.

Notation 6.1.2 *If M is a monoid with identity $e \in M$, we define the partial product $\mathbb{Z}_{\geqslant 0} \widetilde{\times} M$ to be the submonoid of $\mathbb{Z}_{\geqslant 0} \times M$ obtained by removing all pairs $(0, m)$ with $m \neq e$. Here $\mathbb{Z}_{\geqslant 0}$ denotes the monoid of non-negative integers under addition.*

Lemma 6.1.3 *If A is an abelian group, then the Grothendieck group of $\mathbb{Z}_{\geqslant 0} \widetilde{\times} A$ is simply $\mathbb{Z} \times A$ (and the universal map is the obvious inclusion of monoids).*

Proof. Given a morphism of monoids $f : \mathbb{Z}_{\geqslant 0} \widetilde{\times} A \to G$ with G a group, writing $(n, a)(-n+1, 1_A) = (1, a) \in \mathbb{Z} \times A$ we see that the unique extension of f to $\mathbb{Z} \times A$

must be given by

$$\tilde{f}(n,a) = \begin{cases} f(n,a) & n > 0 \\ f(1,a)f(-n+1,1_A)^{-1} & n \leqslant 0. \end{cases}$$

We leave to the reader the (somewhat tedious) verification that $\tilde{f}$ is a homomorphism. (This verification requires the observation that since $\mathbb{Z}_{\geqslant 0}\widetilde{\times}A$ is an abelian monoid, the subgroup of G generated by $f(\mathbb{Z}_{\geqslant 0}\widetilde{\times}A)$ is abelian as well.) □

The idea behind Ho and Liu's obstruction is simply that if we lift the matrices defining a representation up to $\widetilde{U(n)}$, then the relations for G, when applied to the lifts, will produce elements in $\ker(p_n)$. Moreover, paths between representations will lift to paths between such lifts, and this will give path-invariance of the obstruction.

Precisely, the obstruction is defined as follows. Let G be a group with presentation

$$G =< x_1, \ldots, x_g | r_1(x_1, \ldots x_g) = \cdots = r_k(x_1, \ldots, x_g) = 1 >$$

where each r_j is a word in the free group F_g on g letters. Let $N_{j,i}$ denote the total exponent of x_i in r_j (i.e. exponent of x_i in the word $\mathrm{ab}(r_k)$ which is the image of r_k in the abelianization $\mathbb{Z}^g$ of F_g), and let $N_j = \gcd(N_{j,1}, \ldots, N_{j,g})$. (By convention, $\gcd(0, N_1, \ldots, N_l) = \gcd(N_1, \ldots, N_l)$, and $\gcd(0, \ldots, 0) = 0$.)

Consider a representation $\rho : G \to U(n)$ with $\rho(x_i) = A_i$, and choose lifts $(a_i, \alpha_i) \in \widetilde{U(n)}$ of A_i ($i = 1, \ldots, g$). We now define

$$o(\rho) \in \mathbb{Z}/N_1\mathbb{Z} \times \cdots \times \mathbb{Z}/N_k\mathbb{Z}$$

by setting the j^{th} coordinate of $o(\rho)$ to be $o_j(\rho) = [n \sum_{i=1}^{g} N_{j,i} a_i]$ (where the brackets denote reduction modulo N_j).

Remark 6.1.4 *For each j we have*

$$r_j\left((a_1,\alpha_1), \ldots, (a_g,\alpha_g)\right) = \left(\sum_{i=1}^{g} N_{j,i} a_i\,,\ r_j(\alpha_1, \ldots, \alpha_k) \right) \in \ker(p) =< (\frac{1}{n}, e^{-\frac{2\pi i}{n}}) >,$$

so $\sum_{i=1}^{g} N_{j,i} a_i \in \frac{1}{n}\mathbb{Z}$ for each j. Hence the formula defining $o_j(\rho)$ makes sense.

Note that although each $N_{j,i}$ is divisible by N_j, $o_j(\rho)$ need not be trivial in $\mathbb{Z}/N\mathbb{Z}$, because the a_i need not be integers.

Proposition 6.1.5 *The obstruction $o(\rho)$ is well-defined (i.e. does not depend on the lifts chosen) and is an invariant of path components. Moreover, o is additive in the sense that $o(\rho \oplus \psi) = o(\rho) + o(\psi)$. In particular, o defines a morphism of monoids*

$$o : \pi_0(\mathrm{Rep}(G)) \to \mathbb{Z}_{\geqslant 0} \widetilde{\times} \left(\mathbb{Z}/N_1\mathbb{Z} \times \cdots \times \mathbb{Z}/N_l\mathbb{Z}\right),$$

via the formula $o([\rho]) = (\dim(\rho), o(\rho))$.

Proof. Let ρ be a representation $\rho : G \to U(n)$ with $\rho(x_i) = A_i$, and choose lifts $\widetilde{A_i} = (a_i, \alpha_i) \in \widetilde{U(n)}$ for each i. Any other lift of A_i may be written $\widetilde{A}_i' = (y_i/n + a_i, e^{\frac{-2\pi i y_i}{n}}\alpha_i)$ for some $y_i \in \mathbb{Z}$. We now have

$$n\sum_{i=1}^{g} N_{k,i}(a_i + y_i/n) = \left(n\sum_{i=1}^{g} N_{k,i}a_i\right) + \left(\sum_{i=1}^{g} N_{k,i}y_i\right)$$

$$\equiv n\sum_{i=1}^{g} N_{k,i}a_i$$

(modulo $N_k = \gcd(N_{k,i})$). Thus $o(\rho)$ is well-defined.

As noted above, the fact that o is an invariant of path components follows immediately from the fact that paths in $U(n)$ lift to paths in $\widetilde{U(n)}$. The main point here is that the function $n\sum_{i=1}^{g} N_{k,i}a_i$ is continuous as a function of the $a_i \in \mathbb{R}$, but takes values in the discrete set $\mathbb{Z}$ (when restricted to those vectors $(a_1, \ldots, a_g)$ coming from lifts of representations). Hence this function is locally constant.

Finally, we must check that $o(\rho \oplus \psi) = o(\rho) + o(\psi)$ for any $\rho : G \to U(n)$ and $\psi : G \to U(m)$. Say ρ and ψ are represented by matrices $A_i \in U(n)$ and $A_i' \in U(m)$ respectively $(i = 1, \ldots, g)$. Choose lifts $(a_i, \alpha_i) \in \widetilde{U(n)}$ and $(a_i', \alpha_i') \in \widetilde{U(m)}$ of these matrices to the universal covers. Then the homomorphism $\rho \oplus \psi$ is represented by the matrices $A_i \oplus A_i'$ and it is easy to see that

$$\left(\frac{na_i + ma_i'}{n+m},\ e^{-2\pi i \cdot (\frac{na_i + ma_i'}{n+m})}\left((e^{2\pi i a_i}\alpha_i) \oplus (e^{2\pi i a_i'}\alpha_i')\right)\right)$$

is a lift of $A_i \oplus A'_i$ to $\widetilde{U(n+m)}$. Fixing $j \in \{1, \ldots, k\}$, we will check that $o_j(\rho \oplus \psi) = o(\rho) + o(\psi)$. Let $a = \sum_i N_{j,i} a_i$ and let $a' = \sum_i N_{j,i} a'_i$. Then we have $o_j(\rho) = [na]$, $o_j(\psi) = [ma']$ and the above formula gives

$$o_j(\rho \oplus \psi) = \left[(n+m) \sum_i N_{k,i} \left(\frac{na_i + ma'_i}{n+m} \right) \right] = [na + ma'].$$

Hence o is additive. □

In order to obtain concrete results about the path components of representation varieties, we need to look for cases in which the obstruction o is actually a complete invariant of components. In other words, we are interested in finding groups for which the fibers of o are connected.

If, on the other hand, we are only interested in showing the $\mathrm{Rep}(G)$ is stably group-like, we can get away with less. For any $\rho : G \to U(n)$, let $\rho^m : G \to U(mp)$ denote the representation obtained by adding ρ to itself m times. Additivity of o shows that $o(\rho^m) = (0, \ldots, 0)$ for some m (since o takes values in a finite group). Hence if $o^{-1}(0, \ldots 0)$ is connected for infinitely many n, then $\mathrm{Rep}(G)$ is stably group-like.

Fact 6.1.6 *If G is a finitely presented group for which the fibers $o_n^{-1}(0, \ldots, 0)$ of the obstruction maps are connected infinitely often, then* $\mathrm{Rep}(G)$ *is stably group-like.*

Our first class of examples is based on the fundamental group of a (possibly non-orientable) surface S, and the method of proof is an extension of Ho and Liu's computation of the connected components of the space of surface group representations [24, 25]. In particular, the key geometric input is the following theorem of Anton, Malkin and Meinrenken [4, Theorem 7.2] regarding their notion of quasi-Hamiltonian moment maps:

Theorem 6.1.7 *If G is a compact, simply connected Lie group, then any quasi-Hamiltonian moment map $\Phi : M \to G$ has connected fibers.*

An important observation [4, Section 9.2] is that the multiple-commutator map

$\mu_l : G^{2l} \to G$,

$$\mu(a_1, b_1, \ldots, a_l, b_l) = \prod_{i=1}^{l} [a_1, b_i],$$

is a quasi-Hamiltonian moment map. We quickly sketch the proof of this fact. One of the most basic examples of a quasi-Hamiltonian moment map is the double of a group G [4, p. 452]. This is the map $d : G \times G \to G \times G$ which sends (g, h) to $(gh, g^{-1}h^{-1})$. Applying "internal fusion" [4, p. 465] to this moment map shows that the map μ_1 is a q-Hamiltonian moment map. It is a straightforward exercise to check that the cartesian product of two q-Hamiltonian moment maps is still a q-Hamiltonian moment map, and applying fusion repeatedly to the l-fold product of μ_1 with itself yields μ_l. Hence we have:

Corollary 6.1.8 *The l-fold commutator map $\mu_l : G^{2l} \to G$ is a quasi-Hamiltonian moment map. Hence if G is compact and simply connected, μ_l has connected fibers.*

We can now discuss the first class of examples. The proof of the following result traces the arguments of Ho and Liu [24, 25] rather closely.

Theorem 6.1.9 *Let $\omega(c_1, \ldots, c_k) = c_{j_1}^{\nu_1} \cdots c_{j_m}^{\nu_m}$ be a reduced word in the free group on the letters $c_1, \ldots, c_k$, and let G be the group with presentation*

$$\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_k | (\prod_{i=1}^{g} [a_i, b_i]) \omega(c_1, \ldots, c_k) = e \rangle.$$

If $m \leq g$, the monoid of unitary representation spaces $\mathrm{Rep}(G)$ is stably group-like.

If we assume further that for some $j_0 \in \{1, \ldots, k\}$, there is a unique i such that $j_i = j_0$ (in other words, assume that all occurrences of c_{j_0} in ω are adjacent), then $\pi_0(\mathrm{Rep}(G)) \cong \mathbb{Z}_{\geqslant 0} \widetilde{\times} \mathbb{Z}/N\mathbb{Z}$ and $K^0_{\mathrm{def}}(G) \cong \mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, where $N = \gcd(N_1, \ldots N_k)$ and N_j is the total degree of c_j in ω (i.e. $N_j = \sum_{i : j_i = j} \nu_i$).

Finally, if we assume that $N_j = 0$ for each j, then $\pi_0(\mathrm{Rep}(G)) \cong \mathbb{Z}_{\geqslant 0}$ and and $K^0_{\mathrm{def}}(G) \cong \mathbb{Z}$.

Remark 6.1.10 *When $N_j = 0$ for all j, we conclude that the representation variety $\mathrm{Hom}(G, U(n))$ connected for every n. This generalizes the well-known fact that unitary the representation varieties of the fundamental group of a Riemann surface* *are connected. (In the course of the proof, we will also see that* $\mathrm{Hom}(G, SU(n))$ *is connected for any* ω*.)*

The proof of Theorem 6.1.9 can easily be extended to deal with similar groups having multiple relations, but the notation becomes quite cumbersome. Essentially, one may add additional relations of the form $\prod[a_i(\omega'), b_i(\omega')]\omega'(c_1, \ldots, c_k)$*. Here* ω' *is still a word in the generators* c_i*, but the generators* $a_i(\omega')$ *and* $b_i(\omega')$ *are new. As above the number of commutators must be greater than the length (measured as above) of* ω'*. (Such groups may also be constructed as follows: begin with two groups of the form described in Theorem 6.1.9, with generators* a_i, b_i, c_j *and* a_i', b_i', c_j' *respectively. Then form the amalgamated product of these groups by identifying* c_1 *with* c_1'*,* c_2 *with* c_2'*, et cetera. This gives a two-relator group; iterating the process adds more relations.)*

Proof of Theorem 6.1.9. In order to prove that $\mathrm{Rep}(G)$ is stably group-like, it suffices (by Fact 6.1.6), to check that $o^{-1}([0])$ is always connected. In the cases where we are able to determine the complete structure of $\pi_0\mathrm{Rep}(G)$, we will prove that the obstruction o is a complete invariant for the path components of $\mathrm{Hom}(G, U(n))$, i.e. that $o^{-1}([r])$ is connected for any $r \in \mathbb{Z}$. This will show that the obstruction map gives an injection $\pi_0\mathrm{Rep}(G) \hookrightarrow \mathbb{Z}/N\mathbb{Z}$. The desired computations will then follow easily.

To study the obstruction map, we follow [25] and define

$$X_{U(n)}(Y) = \{A_1, B_1, \ldots, A_g, B_g, C_1, \ldots C_k \mid \big(\prod_{i=1}^{g}[A_i, B_i]\big)\omega(C_1, \ldots, C_k) = Y\}.$$

The space $X_G(Y)$ is defined similarly for any group G; note that when $Y = I$ we obtain the representation spaces.

Call the word ω *good* if it satisfies the hypothesis for the second part of the theorem; without loss of generality we may assume in this case that there is a unique i for which $j_i = 1$. We will mainly work in the case when ω is good, and we will show that in this case $o^{-1}([r])$ is connected for any $r \in \mathbb{Z}$. The first and last statements of the theorem essentially follows from the observation that the argument does not require ω to be good if $r = 0$. We will deal with all three cases simultaneously in what follows, pointing out the places where the arguments differ.

To show that $o^{-1}([r])$ is connected (for some fixed $r \in \mathbb{Z}$), let $\zeta_n = (1/n, e^{\frac{-2\pi i}{n}})$ denote our chosen generator for $\ker(p_n)$, and consider the space $X_{\widetilde{U(n)}}(\zeta_n^r)$. The map p induces a map $X_{\widetilde{U(n)}}(\zeta_n^r) \to o^{-1}([r])$. We will show that this map is surjective, and that $X_{\widetilde{U(n)}}(\zeta_n^r)$ is connected if either ω is good or $r = 0$. As we will see, these statements essentially suffice to prove the theorem. For surjectivity, consider any $\rho = (\{A_i\}, \{B_i\}, \{C_j\}) \in o^{-1}([r])$. Then there exist lifts $\widetilde{A}_i$, $\widetilde{B}_i$, and $\widetilde{C}_j$ in $\widetilde{U(n)}$ such that

$$\Big(\prod_{i=1}^{g} [\widetilde{A}_i, \widetilde{B}_i]\Big)\omega(\widetilde{C}_1, \ldots, \widetilde{C}_k) = \zeta_n^r \cdot \zeta_n^{mN}$$

for some $m \in \mathbb{Z}$. Since $N = \gcd(N_1, \ldots, N_j)$ we may write

$$mN = \sum_j m_j N_j$$

for some $m_j \in \mathbb{Z}$. Let

$$\widetilde{C}_j' = \widetilde{C}_j \cdot \zeta_n^{-m_j}.$$

Since $\zeta_n^{-m_j}$ is central in $\widetilde{U(n)}$, we see that

$$\Big(\prod_{i=1}^{g} [\widetilde{A_i}, \widetilde{B_i}]\Big)\omega(\widetilde{C}_1', \ldots, \widetilde{C}_k') = \zeta_n^r \cdot \zeta_n^{mN} \cdot \zeta_n^{\sum -m_j N_j} = \zeta_n^r.$$

This proves surjectivity, since $(\{\widetilde{A}_i\}, \{\widetilde{B}_i\}, \{\widetilde{C}_j'\}) \in X_{\widetilde{U(n)}}(\zeta_n^r)$ and $p(\widetilde{C}_j') = C_j$.

Next, we consider connectivity of $X_{\widetilde{U(n)}}(\zeta_n^r)$. We can write $X_{\widetilde{U(n)}}(\zeta_n^r)$ as

$$\begin{aligned} X_{\widetilde{U(n)}}(\zeta_n^r) \quad = \quad & \{(a_1, \alpha_1), \ldots, (b_g, \beta_g), (c_1, \gamma_1), \ldots, (c_k, \gamma_k) \in \widetilde{U(n)} \mid \\ & \sum_j N_j c_j = \frac{r}{n} \text{ and } \prod_i [\alpha_i, \beta_i] = e^{\frac{-2\pi i r}{n}} I\}. \end{aligned} \tag{6.1}$$

Assuming ω is good, we have $N_1 = \nu_1 \neq 0$ and hence

$$X_{\widetilde{U(n)}}(\zeta_n^r) \cong \mathbb{R}^{2l+k-1} \times X_{SU(n)}(e^{-2\pi i r/n} I).$$

Hence when ω is good it will suffice to show that $X_{SU(n)}(e^{-2\pi i r/n} I)$ is connected for any $r \in \mathbb{Z}$. To prove the first statement in the theorem, we need only show that $X_{SU(n)}(\zeta_n^0) = \mathrm{Hom}(G, SU(n))$ is connected, and this also suffice for the third statement because when $N_j = 0$ for all j, formula (6.1) shows that $X_{SU(n)}(e^{-2\pi ir/nI})$ is empty unless $r = 0$ (so in particular all representations have trivial obstruction class), in which case this space is homeomorphic to

$$\mathbb{R}^{2g+k} \times X_{SU(n)}(I) = \mathbb{R}^{2g+k} \times \mathrm{Hom}(G, SU(n)).$$

Consider the map $Q : X_{SU(n)}(e^{2\pi ir/n}I) \to SU(n)^k$ given by projecting onto the last k factors. The fiber $Q^{-1}(\gamma_1, \ldots, \gamma_k)$ is homeomorphic to the fiber of the multiple-commutator map $\mu_g : SU(n)^{2g} \to SU(n)$ over the point $\omega(\gamma_1, \ldots, \gamma_k)$ and hence is connected by Corollary 6.1.8. Thus to show that $X_{SU(n)}(e^{-2\pi ir/n}I)$ is connected, it will suffice to produce paths between the fibers of Q.

When ω is good, we will produce a path starting in the fiber over

$$z_r = (q_r, I, \ldots, I)$$

and ending in the fiber over $(\gamma_1, \ldots, \gamma_k)$, where $q_r \in SU(n)$ is a diagonal matrix such that $q_r^{N_1} = e^{\frac{2\pi ir}{n}} I$ (since the diagonal subgroup of $SU(n)$ is a torus, such an N_1^{th}-root always exists; it is of course easy to write it out explicitly). In the other cases, we may assume $r = 0$, and we simply set $q_r = I$ and use the argument to follow.

To construct the desired path, we begin by choosing $g_j \in SU(n)$ such that $\gamma_j' := g_j^{-1}\gamma_j g_j$ is diagonal $(j = 1, \ldots, k)$. Also, let $g_j(t)$ be a path in $SU(n)$ with $g_j(0) = I$ and $g_j(1) = g_j$. Letting $T \subset SU(n)$ denote the diagonal torus, we can choose $\xi_j \in \mathrm{Lie}(T)$ such that $\exp(\xi_j) = \gamma_j'$ for $j = 2, \ldots, k$ and $\exp(\xi_1) = \gamma_1' q_r^{-1}$. We now define paths

$$\gamma_j(t) = g_j(t) \exp(t\xi_j) g_j(t)^{-1} \ \ (j = 1, \ldots, k)$$

in $SU(n)$.

We claim that there exist paths $a_l(t)$ and $b_l(t)$ $(l = 1, \dots, m)$ such that

$$[a_l, b_l] = \gamma_{j_l}(t)^{-\nu_l}. \tag{6.2}$$

Assuming this for the moment, we complete the proof of connectivity. Consider the path

$$(I, \dots, I, a_m(t), b_m(t), \dots, a_1(t), b_1(t), \gamma_1(t)g_1(t)q_r g_1(t)^{-1}, \gamma_2(t), \dots, \gamma_k(t)).$$

It is easy to check that this path starts in $Q^{-1}(q_r, I, \dots, I)$, ends in $Q^{-1}(\gamma_1, \dots, \gamma_k)$. In fact, this path lies in $X_{SU(n)}(e^{-2\pi ir/n}I)$ for each $t \in [0,1]$ because

$$(\gamma_1(t)g_1(t)q_r g_1(t)^{-1})^{N_1} = \gamma_1(t)^{N_1} g_1(t) q_r^{N_1} g_1(t)^{-1} = \gamma_1(t)^{N_1} \cdot e^{\frac{2\pi ir}{n}} I$$

and $e^{\frac{2\pi ir}{n}} I$ is central in $SU(n)$. Hence

$$\omega(I, \dots, I, a_m(t), b_m(t), \dots, a_1(t), b_1(t), \gamma_1(t)g_1(t)q_r g_1(t)^{-1}, \gamma_2(t), \dots, \gamma_k(t))$$

$$= e^{\frac{2\pi ir}{n}} I \cdot [a_m(t), b_m(t)] \cdots [a_1(t), b_1(t)] \gamma_{j_1}(t)^{\nu_1} \cdots \gamma_{j_m}(t)^{\nu_m}$$

and by (6.2), everything but $e^{\frac{2\pi ir}{n}} I$ cancels. Note that in the construction of this path we have used the assumption $m \leqslant g$, and that it is important that c_1 appear only once in ω, since otherwise we would have non-central matrices $q_r^{\nu_l}$ appearing for those l with $j_l = 1$.

The construction of the paths a_i and b_i follows [24]; for completeness we repeat the argument. Let $D \subset SU(n)$ denote the diagonal torus and choose a Coxeter element $w = \alpha D$ in the Weyl group $N(D)/D$. When w acts by conjugation on Lie(D), none of its eigenvalues equal 1 [26]. Identifying Lie(D) with a subspace of $n \times n$ matrices, we see that the map $\xi \mapsto \alpha\xi\alpha^{-1} - \xi$ is onto, and hence we may choose ξ_j' such that $\xi_j = \alpha\xi_j'\alpha^{-1} - \xi_j'$ for $j = 1, \dots, k$. Multiplying by s and exponentiating gives

$$\alpha \exp(s\xi_j')\alpha^{-1} \exp(-s\xi_j') = \exp(s\xi_j')$$

for any $s \in \mathbb{R}$. Now, setting $a_i(t) = g_{j_i} \alpha g_{j_i}^{-1}$ (a constant path) and

$$b_i(t) = g_{j_i} \exp(-a_i t \xi_i') g_{j_i}^{-1},$$

one easily checks that (6.2) holds. This completes the proof that $X_{SU(n)}(e^{2\pi i r/n} I)$ is connected for all r when ω is good, and also completes the proof that $X_{SU(n)}(I)$ is connected for any ω. Thus we have shown that $\mathrm{Rep}(G)$ is stably group-like for any ω, and that o is a complete invariant when either $N_j = 0$ for all j or ω is good.

Finally, we must determine the structure of $\pi_0(\mathrm{Rep}(G))$ and $K^0_{\mathrm{def}}(G)$. When $N = 0$, we have shown that all representations have trivial obstruction class and that $\mathrm{Hom}(G, U(n))$ is connected. Hence $\pi_0 \cong \mathbb{Z}_{\geqslant 0}$ and $K^0_{\mathrm{def}}(G) \cong \mathbb{Z}$ in this case. When ω is good, o defines a morphism $o : \pi_0(\mathrm{Rep}(G, U(n))) \to \mathbb{Z}_{\geqslant 0} \widetilde{\times} \mathbb{Z}/N\mathbb{Z}$. This map is injective because o is a complete invariant of path components. So we just need to check that it is surjective, i.e. we need to produce representations in each obstruction class. Note that it will suffice to do this for $n = 1$, since o is a homomorphism and hence block sum with the identity does not change the obstruction class. Again we assume $N_1 \neq 0$. Given $r \in \mathbb{Z}$, let ρ_r be the representation defined by setting $A_i = B_i = 1$ $(i = 1, \ldots l)$, $C_1 = e^{\frac{2\pi i r}{N_1}}$ and $C_j = 1$ for $j = 2, \ldots, k$. Setting $\widetilde{A}_i = \widetilde{B}_i = (0, 1)$, $\widetilde{C}_1 = (\frac{r}{N_1}, 1)$ and $\widetilde{C}_j = (0, 1)$ for $j > 1$, we see that $o(\rho) = [r]$ as desired. Since $K^0_{\mathrm{def}}(G)$ is the Grothendieck group of $\pi_0(\mathrm{Rep}(G))$ (Lemma 2.0.5), Lemma 6.1.3 shows that $K^0_{\mathrm{def}}(G) = \mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$. □

We now present a second class of examples for which $\mathrm{Rep}(G)$ is stably group-like, and again we will show that the obstruction o gives a complete invariant of path components. Falling into this class is the group $\langle a, b \mid a^2 = b^2 \rangle$, which is the fundamental group of $\mathbb{R}P^2 \# \mathbb{R}P^2$. This group is missing from the previous class of examples, since no commutators appear in the presentation, and is also not covered by the results of [24, 25] (for the same reason).

Proposition 6.1.11 *Let $G_{k,m}$ denote the group with presentation*

$$G_{k,m} = \langle x_1, x_2, \ldots, x_k | x_1^m = x_2^m = \cdots = x_k^m \rangle.$$

Then the monoid of unitary representation spaces $\mathrm{Rep}(G_{k,m})$ *is stably group-like,* *and the obstruction map induces an isomorphism*

$$\pi_0(\mathrm{Rep}(G_{k,m})) \cong \mathbb{Z}_{\geqslant 0}\widetilde{\times}(\mathbb{Z}/m\mathbb{Z})^{k-1}.$$

It follows that

$$K^0_{\mathrm{def}}(G_{k,m}) \cong \mathbb{Z} \times (\mathbb{Z}/m\mathbb{Z})^{k-1}.$$

Proof. We begin by noting that the computation of $K^0_{\mathrm{def}}(G_{k,m})$ follows immediately from the computation of $\pi_0(\mathrm{Rep}(G))$ (as in the proof of Theorem 6.1.9) so we proceed to analyze the latter.

In order to use Proposition 6.1.5, we need to specify the relations for $G_{k,m}$: we will use the relations $r_i = x_i^m x_{i+1}^{-m}$ for $i = 1, \ldots, k-1$. It will suffice (by Proposition 6.1.5) to show that $o^{-1}([l_1], \ldots, [l_k])$ is connected for any $l_i \in \mathbb{Z}$, and that representations of each obstruction class exist. The latter statement is immediate, since it suffices by additivity to produce representations into $U(1)$ with each obstruction class, and one can set $x_1 = e^{\frac{2\pi i l_1}{m}}$ and $\tilde{x_1} = l_1/m \in \widetilde{U(n)} = \mathbb{R}$, and then solve recursively $\tilde{x_2}, \ldots, \tilde{x_k} \in \widetilde{U(1)} = \mathbb{R}$, giving the desired result.

Now we must show that the fibers of o are connected. Since there exist representations in each obstruction class, it suffices to show that $\mathrm{Hom}(G_{k,m}, U(n))$ has at most m^{k-1} components. Given any representation $\rho : G_{k,m} \to U(n)$, we will show that ρ is connected by a path in $\mathrm{Hom}(G_{k,m}, U(n))$ to a representation of the form

$$x_1 \mapsto e^{\frac{2\pi i l_1}{m}} \oplus I_{n-1}, \ldots, x_{k-1} \mapsto e^{\frac{2\pi i l_{k-1}}{m}} \oplus I_{n-1}, x_k \mapsto I_n$$

(here the l_i are integers between 0 and $m-1$). Since there are m^{k-1} such representations, we obtain the desired bound on $|\pi_0\mathrm{Hom}(G_{k,m}, U(n))|$.

Let $\rho(x_i) = A_i$. Note that since $U(n)$ is connected, any two conjugate representations lie in the same path component. Hence we may assume that A_k is a diagonal matrix. Now write

$$A_k^m = \lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_l I_{n_p}$$

for some distinct $\lambda_j \in S^1$ and some partition $(n_1, \ldots, n_p)$ of n. Now, since $A_i^m = A_k^m$, each eigenvalue of A_i is an m^{th} root of some λ_j. If $v \in \mathbb{C}^n$ is an eigenvector of A_i, we have $A_i v = \sqrt[m]{\lambda_j} v$ for some j, and now we have $A_k^m v = A_i^m v = (\sqrt[m]{\lambda_j})^m v = \lambda_j v$, meaning that v lies in the λ_j-eigenspace of A. It is now easy to see that in the standard basis for $\mathbb{C}^n$, each A_i has a block decomposition

$$A_i = A_{i,1} \oplus \cdots \oplus A_{i,p}$$

with $A_{i,j} \in U(n_j)$. Moreover, $A_{i,j}^m = \lambda_j I_{m_j}$, so $A_{i,j}$ is determined by its flag of eigenspaces for the various m^{th} roots of λ_j. But the for any fixed elements $a_1, \ldots, a_m \in S^1$ and fixed partition $j = j_1 + \cdots + j_m$, the space of matrices A in $U(j)$ such that A has an a_1-eigenspace of dimension j_1, etc., is a flag manifold, hence connected (more precisely, this space is homeomorphic to $U(j)/\prod_i U(j_i)$. Moreover, this subspace certainly contains diagonal matrices. Now, replacing the $A_{i,j}$ with any matrices in these subspaces produces a representation, and so we may connect our representation ρ to a representation ρ' in which each matrix is diagonal.

Next, note that there is a path D_t of diagonal matrices from the diagonal matrix $\rho'(x_k)^{-1}$ to the identity. Since all the matrices in question are diagonal, they commute with D_t, and multiplying through by D_t yields a path of representations ρ'_t connecting ρ' to a representation ρ'' with $\rho''(x_i)$ diagonal and $\rho''(x_k) = I$.

To finish the proof, we will check that the representations $\rho \in \mathrm{Hom}(G_{k,m}, U(2))$ given by

$$\rho(x_i) = A_i = \zeta_m^{a_i} \oplus \zeta_m^{b_i} \ (i = 1, \ldots, k-1)\ , \rho(x_k) = I_2$$

and $\psi \in \mathrm{Hom}(G_{k,m}, U(2))$ given by

$$\psi(x_i) = A_i \text{ for } i \neq i_0 \text{ and } \psi(x_{i_0}) = \zeta_m^{a_{i_0}+b_{i_0}} \oplus 1$$

are connected by a path; here $\zeta_m = e^{2\pi i/m}$ and $1 \leqslant i_0 \leqslant k-1$. (The full result then follows by an easy induction, in which one sets the last entry of each matrix to 1, the the next to last, and so on.) First, we can multiply through by a path of diagonal matrices starting at I_2 and ending at $1 \oplus \zeta_m^{a_{i_0}}$. This yields a path from ρ to the representation ρ' given by

$$\rho'(x_i) = \zeta_m^{a_i} \oplus \zeta_m^{b_i + a_{i_0}} \ (i < k),\ \rho(x_k) = 1 \oplus \zeta_m^{a_{i_0}}.$$

Now, if we choose a path in $\mathbb{C}P^1 = S^2$ from the $\zeta_m^{a_{i_0}}$-eigenspace of $\rho'(x_{i_0})$ to its orthogonal complement, we get a path of m^{th}-roots of I_2 starting at $\zeta_m^{a_{i_0}} \oplus \zeta_m^{b_{i_0}+a_{i_0}}$ and ending at $\zeta_m^{a_{i_0}+b_{i_0}} \oplus \zeta_m^{a_{i_0}}$. This yields a path of representations from ρ' to the representation ρ'' given by

$$\rho''(x_{i_0}) = \zeta_m^{a_{i_0}+b_{i_0}} \oplus \zeta_m^{a_{i_0}}, \quad \rho''(x_i) = \rho'(x_i) \text{ for } i \neq i_0.$$

Multiplying through by a path from I_2 to $1 \oplus \zeta_m^{a_{i_0}}$ gives a path from ρ'' to ψ, completing the proof. □

6.2 Finitely generated abelian groups

We now discuss an example in which the groups in question are stably group-like with respect to larger representations (rather than the representation 1). This result applies to both unitary and general linear deformation K-theory. First we need a lemma regarding simultaneous diagonalizability of commuting matrices.

Lemma 6.2.1 *Let $A_1, \ldots, A_k \in U(n)$ be commuting unitary matrices. Then there is a matrix $X \in U(n)$ such that for $i = 1, \ldots, k$, the matrix XA_iX^{-1} is diagonal. The same statement holds with $U(n)$ replaced by $GL_n(\mathbb{C})$.*

Proof. It is well-known that commuting diagonalizable operators admit a basis consisting of simultaneous eigenvectors (for a short proof, see [11]). This completes the proof for $GL_n(\mathbb{C})$. For the unitary case, we need to show that this basis can be assumed orthonormal, since then the matrix X taking this basis to the standard basis is unitary. Say $\{v_1, \ldots, v_n\}$ is a basis for $\mathbb{C}^n$ such that $A_iv_j = \lambda_{ij}v_j$ for some $\lambda_{i,j} \in S^1$. Replacing v_j by $v_j/|v_j|$ if necessary, we may assume each v_j has length one. We will now describe an inductive procedure for making this basis orthogonal.

Assume that the first $l-1$ basis vectors are orthogonal to all the other v_i, i.e. assume that for $1 \leqslant j \leqslant l-1$ and $i \neq j$ we have $\langle v_i, v_j\rangle = 0$. (Here we allow $l = 1$.) Consider the set $Z = \{j : \lambda_{ij} = \lambda_{il} \text{ for all } i\}$. Observe that Z contains l, and is hence non-empty. It is easy to check that

$$\text{Span}\{v_j : j \in Z\} = \{w : A_iw = \lambda_{il}w \text{ for all } i\}.$$

Let $\{w_j : j \in Z\}$ be an orthonormal basis for $\mathrm{Span}\{v_j : j \in Z\}$. It follows from the definitions that $\mathcal{B} = \{v_j : j \notin Z\} \cup \{w_j : j \in Z\}$ is a simultaneous eigenbasis for the A_i. The set $\mathcal{B}$ is ordered (the ordering on the w_j is of course arbitrary), and we claim that it satisfies the induction hypothesis with $l-1$ replaced by l. This will complete the induction step, and the proof. If b is one of the first l elements in $\mathcal{B}$, then either $b = v_j$ with $j < l$ and $j \notin Z$, or $b = w_j$ with $j \leqslant l$ and $j \in Z$.

In the first case, we know by assumption that v_j is orthogonal to all $v_{j'}$ $(j \neq j')$ so we just need to check that v_j is orthogonal to $w_{j'}$. But since $j \notin Z$, there exists i such that $\lambda_{ij} \neq \lambda_{il}$, i.e. v_j is not in the λ_{il}-eigenspace of A_i. Since A_i is unitary, this means v_j is orthogonal to this entire eigenspace. By construction, $w_{j'}$ lies in this eigenspace and so $\langle v_j, w_{j'}\rangle = 0$.

In the second case, we know by assumption that w_j is orthogonal to $w_{j'}$ for each $j' \in Z$, so we just need to check that w_j is orthogonal to $v_{j'}$ for $j' \notin Z$. But this is precisely what was proven in the previous case. □

Proposition 6.2.2 *Let $A = \mathbb{Z}^k \times T$ be a finitely generated abelian group, with T the torsion subgroup of A. Let $\rho_1, \cdots, \rho_m : T \to U(1)$ be the irreducible characters of T. Then* $\mathrm{Rep}(A)$ *is stably group-like with respect to*

$$\rho = \bigoplus_i \rho_i.$$

(Here the ρ_i are extended to A by the projection $A \to T$.)

Proof. We claim that the components of the representations ρ_i $(i = 1, \ldots, m)$ generate $\pi_0\mathrm{Rep}(A)$. This implies A is stably group-like with respect to ρ, as in Example 3.0.9. Let $\alpha \in \mathrm{Hom}(A, U(n))$ be any representation of A. Then α is determined by a representation α' of T together with matrices $X_1, \ldots, X_k \in \mathrm{Stab}(\alpha')$. Now, the collection of matrices $\{X_1, \ldots, X_k\} \cup \{\alpha'(a) | a \in T\}$ are pairwise commuting, so by Lemma 6.2.1 these matrices are simultaneously diagonalizable by a unitary matrix S. Choosing a path from S to the identity gives a path from α to a representation lying in the diagonal subgroup of $U(n)$. Moreover, we may now choose paths in the diagonal connecting the images of the free generators of A to the identity, so we conclude that α is connected to a representation $\widetilde{\alpha}$ which factors through T. The representation $\widetilde{\alpha}$ is a sum of characters, and hence $\pi_0(\mathrm{Rep}(A))$ is generated by the representations ρ_i, as claimed. $\square$

Remark 6.2.3 *The above argument provides a non-standard proof that irreducible representations of abelian groups are one-dimensional. The same argument appears in [11].*

Appendix A

Holonomy of flat connections

The goal of this appendix is to give a careful discussion of the holonomy representation associated to a flat connection on a principal G-bundle over a connected manifold M. We show that holonomy induces a bijection from the set of all such (smooth) connections to the set of representations of $\pi_1 M$ into G, after taking the action of the based gauge group into account (Theorem A.0.20). This material is essentially well-known, but there does not appear to be any published reference. Some of the results to follow may be found in Morita's books [36, 37], and the main result is essentially stated in the introduction to [14].

All manifolds and maps in this appendix will be smooth.

Definition A.0.4 *Let G be a Lie group. A principal G-bundle P over a manifold M is a manifold P together with a map*

$$\pi : P \to M$$

such that there exist local trivializations $P|_U \cong U \times G$, and such that the transition maps $(U \cap V) \times G \xrightarrow{\cong} (U \cap V) \times G$ all have the form $(x, g) \mapsto (x, h(x) \cdot g)$ for some smooth map $h : U \cap V \to G$. Note that with this definition, P acquires a right *action of the structure group G.*

Let $\pi : P \to M$ be a principal G-bundle. Then there is a natural map of vector bundles over P

$$TP \xrightarrow{\alpha} \pi^* TM$$

which is equivariant with respect to the action of G.

Definition A.0.5 *A connection on P is a G-equivariant splitting of the map α, i.e. a (smooth) map of vector bundles*

$$A : \pi^* TM \longrightarrow TP$$

such that $\alpha \circ A = id_{\pi^* TM}$. We denote the set of all connections on P by $\mathcal{A}(P)$.

Next we introduce the action of the gauge group $\mathcal{G} = \mathcal{G}(P)$ on the set $\mathcal{A}(P)$ of connections.

Definition A.0.6 *The gauge group $\mathcal{G}(P)$ is the group of all equivariant maps $P \xrightarrow{\phi} P$ such that $\pi \circ \phi = \pi$.*

Given a map $P \xrightarrow{\phi} Q$ of principal G-bundles over M (that is, an equivariant map such that $\pi_2 \circ \phi = \pi_1$, where $\pi_1 : P \to M$ and $\pi_2 : Q \to M$ are the projections) we obtain a map $\phi_* : \mathcal{A}(P) \to \mathcal{A}(Q)$ as follows: given a connection A on P, consider the diagram

$$\begin{tikzcd}
TP \arrow[r] \arrow[d, "D\phi"'] & \pi_1^* TM \arrow[l, bend right, "A"'] \\
TQ \arrow[r] & \pi_2^* TM, \arrow[u, "\widetilde{\phi}^{-1}"'] \arrow[l, dashed, "\phi_* A"']
\end{tikzcd}$$

where $\widetilde{\phi}^{-1}$ denotes the natural map induced from ϕ^{-1}; note that $\widetilde{\phi}$ is invertible because ϕ is invertible (all principal bundle maps are diffeomorphisms). Then we define $\phi_* A$ to be the map

$$D\phi \circ A \circ \tilde{\phi}^{-1}.$$

It is straightforward to check that $\phi^* A$ is an equivariant splitting.

Definition A.0.7 *The (left) action of $\mathcal{G}(P)$ on $\mathcal{A}(P)$ is given by $(A, \phi) \mapsto \phi_* A$.*

We are now ready to discuss parallel transport on a principal G-bundle P equipped with a connection A, which will lead to the holonomy representation. Given a smooth curve $\gamma : [0,1] \to M$ (which we assume extends smoothly to $\mathbb{R}$ and is constant outside $(-\epsilon, 1+\epsilon)$ for some $\epsilon > 0$) we want to define a parallel transport operator $T_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$, where $P_{\gamma(0)}$ and $P_{\gamma(1)}$ denote the fibers of P over these points.

This operator is defined by following A-horizontal lifts of the path γ. Specifically, we have a vector field V_γ on the pullback $\gamma^* P$, defined by

$$V_\gamma(t,p) = (\varepsilon_t, A(\gamma'(t), p))$$

for any $t \in (-\epsilon, 1+\epsilon)$, $p \in P_{\gamma(t)}$. Here ε_t denotes the canonical unit vector in $T_t(-\epsilon, 1+\epsilon)$; so $\gamma'(t) = (D\gamma)(\varepsilon_t)$. This is a smooth vector field on $\gamma^* P$, and hence for any $p \in P_{\gamma(0)}$ there exists a unique curve $\Gamma_p : \mathbb{R} \to \gamma^* P$ with $\Gamma_p'(t) = V_\gamma\left(\Gamma_p(t)\right)$ and $\Gamma_p(0) = (0,p) \in \gamma^* P$. Our horizontal lift of γ, starting at $p \in P_{\gamma(0)}$, will be the curve $\tilde{\gamma}_p = f \circ \Gamma_p|_{[0,1]}$, where $f : \gamma^* P \to P$ is the natural map.

Lemma A.0.8 *The curve $\tilde{\gamma}_p : [0,1] \to P$ satisfies*

$$\pi \circ \tilde{\gamma}_p = \gamma \quad \textit{and} \quad \tilde{\gamma}_p'(t) = A\left(\gamma'(t), \tilde{\gamma}_p(t)\right)$$

Proof. We may write $\Gamma_p : \mathbb{R} \to \gamma^* P$ as $\Gamma_p = (\Gamma_1, \Gamma_2)$ with $\Gamma_1 : \mathbb{R} \to \mathbb{R}$ and $\Gamma_2 : \mathbb{R} \to P$ (so $\Gamma_2 = \tilde{\gamma}_p$). Now, the equation $\Gamma_p'(t) = V_\gamma(\Gamma_p(t))$ becomes

$$\left(\Gamma_1'(t), \tilde{\gamma}_p'(t)\right) = \left(\varepsilon_{\Gamma_1(t)}, A\left(\gamma'(\Gamma_1(t)), \tilde{\gamma}_p(t)\right)\right),$$

from which we see that $\Gamma_1(t)$ satisfies the differential equation $\Gamma_1'(t) = \varepsilon_{\Gamma_1(t)}$. This equation is also satisfied by the identity map $\mathbb{R} \to \mathbb{R}$ (by definition of ε) so $\Gamma_1(t) = t$ for any $t \in \mathbb{R}$.

We now have

$$\tilde{\gamma}_p'(t) = A\left(\gamma'(\Gamma_1(t)), \tilde{\gamma}_p(t)\right) = A\left(\gamma'(t), \tilde{\gamma}_p(t)\right)$$

as desired, and

$$D\pi(\tilde{\gamma}_p'(t)) = D\pi\left(A(\gamma'(t), \Gamma_2(t))\right) = \gamma'(t)$$

(because A is a splitting) which implies $\pi \circ \tilde{\gamma}_p(t) = \gamma(t)$. □

Definition A.0.9 *Given a connection A on a principal G-bundle P over M, the parallel transport operator associated to a smooth curve $\gamma : [0,1] \to M$ is the function*

$$T_\gamma = T_\gamma^A : P_{\gamma(0)} \to P_{\gamma(0)}$$

defined by $T_\gamma(p) = \tilde{\gamma}_p(1)$.

We now record several simple properties of parallel transport, which are easily checked.

Lemma A.0.10 *For any $\gamma : I \to M$, $p \in P_{\gamma(0)}$, and $g \in G$,*

a) $(\tilde{\gamma}_p) \cdot g = \tilde{\gamma}_{p \cdot g}$;

b) Parallel transport is G-equivariant, i.e.

$$T_\gamma(p \cdot g) = T_\gamma(p) \cdot g;$$

c) $T_{\overline{\gamma}} = T_\gamma^{-1}$, *where* $\overline{\gamma}(t) = \gamma(1-t)$;

d) $T_{\alpha \centerdot \gamma} = T_\gamma \circ T_\alpha$, *if* $\alpha : I \to M$ *satisfies* $\alpha'(1) = \gamma'(0)$;

e) If γ is constant, then $T_\gamma = \mathrm{Id}$.

Here we have used the notation $p_1 \centerdot p_2$ for composition of paths (tracing out p_1 first).

When our connection A is *flat*, parallel transport will be homotopy invariant, and will allow us to define the *holonomy representation* $\rho_A : \pi_1 M \to G$. We will need the following result from [44, p. 349], which may be taken as the definition of flatness, for the purposes of this appendix.

Proposition A.0.11 *Let $P \xrightarrow{\pi} M$ be a principal G-bundle and let A be a flat connection on P. Then for each $m \in M$ there exists a neighborhood $U \ni m$ on which A is isomorphic to the trivial connection; that is, there exists a map of principal G-bundles $\pi^{-1}(U) \xrightarrow{t} U \times G$ such that $A|_U = t^*\tau$, where τ is the trivial connection on $U \times G$ (so $\tau((U,g),V) = (V,0) \in T_uU \times T_gG$). Here t^* denotes the pullback operator on connections, defined analogously to the pushforward (so $A|_U = t^*\tau \Leftrightarrow t_*(A|_U) = \tau$).*

Lemma A.0.12 *Let A be a flat connection on P, and let $\gamma_0, \gamma_1 : [0,1] \to M$ be two paths. If γ_0 is homotopic to γ_1 (relative to $\{0,1\}$) then the parallel transport operators T_{γ_0} and T_{γ_1} are equal.*

Proof. To begin, note that if τ is the trivial connection on $M \times G$, then the τ-horizontal lifts of any curve $\gamma : I \to M$ are simply $\tilde{\gamma}_{(m,g)}(t) = (\gamma(t), g)$. This implies that T^τ_γ is always the identity, and moreover if $t = (\pi, t_2) : P \to M \times G$ is a map of principal bundles, then the $t^*\tau$-horizontal lift of $\gamma : I \to M$ (at $p \in P$) is just $t^{-1}(\gamma, t_2(p))$. Hence the map $T^{t^*\tau}_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$ is

$$T^{t^*\tau}_\gamma(p) = t^{-1}(\gamma(1), t_2(p)),$$

and is independent of the path γ.

Now, say A is a flat connection on P and $H : I \times I \to M$ is a homotopy between $H(t,0) = \gamma_0$ and $H(t,1) = \gamma_1$ (with $H|_{\{0\}\times I}$ and $H|_{\{1\}\times I}$ constant). We may assume H is smooth (and extends smoothly over $(-\epsilon, 1+\epsilon)^2$). By Proposition A.0.11 and compactness, we know that there exists a finite covering of $I \times I$ by open sets $\{U_i\}_{i=1}^n$ over which A is trivial. In particular we may assume that for some $\varepsilon > 0$ and some $0 < t_0 < t_1 < ... < t_k = 1$ we have $[t_{i-1}, t_i] \times [0, \varepsilon] \subset U_i$ $(i = 1, ..., k)$. Now parallel transport along $\gamma_0|_{[t_{i-1},t_i]}$ agrees with parallel transport along the other three sides of the square $[t_{i-1}, t_i] \times [0, \varepsilon]$, and by induction we see (using Lemma A.0.10 c), d) and e)) that $T_{\gamma_0} = T_{\gamma_\varepsilon}$. We can complete the proof by iterating this process. The simplest way to see that this process terminates at γ_1 is probably to assume that U_i are circles, in which case the supremum of the ε which may be used above is the second coordinate of some intersection point between these circles. The same is true at each stage of the process and there are only finitely many such intersection points. □

We can now define the holonomy representation.

Definition A.0.13 *Let P be a principal G-bundle over M, and choose basepoints $m_0 \in M$, $p_0 \in P_{m_0}$. Associated to any flat connection A on P, the* holonomy representation

$$\rho_A : \pi_1(M, m_0) \to G$$

is defined by setting $\rho_A([\gamma])$ to be the unique element of G satisfying

$$p_0 = T^A_\gamma(p_0) \cdot \rho_A([\gamma]).$$

Here $\gamma : I \to M$ is a smooth loop based at m_0 and $[\gamma]$ is its class in $\pi_1(M, m_0)$.

Note here that $T^A_\gamma : P_{m_0} \to P_{m_0}$ depends only on $[\gamma]$ (by Lemma A.0.12) and that ρ_A is a homomorphism:

$$\begin{aligned} T^A_{\gamma_1 \centerdot \gamma_2}(p_0) \cdot (\rho_A([\gamma_1])\rho_A([\gamma_2])) &= T^A_{\gamma_2}\left(T^A_{\gamma_1}(p_0)\right) \cdot \rho_A([\gamma_1])\rho_A([\gamma_2]) \\ &= T^A_{\gamma_2}\left(T^A_{\gamma_1}(p_0) \cdot \rho_A([\gamma_1])\right) \cdot \rho_A([\gamma_2]) \\ &= T^A_{\gamma_2}(p_0) \cdot \rho_A([\gamma_2]) \\ &= p_0, \end{aligned}$$

so $\rho_A([\gamma_1])\rho_A([\gamma_2]) = \rho_A([\gamma_1 \centerdot \gamma_2])$.

From here on we assume that M is equipped with a basepoint $m_0 \in M$, and we assume all principal bundles P are equipped with basepoints $p_0 \in P_{m_0}$.

We now describe how holonomy changes as we vary the basepoints $p_0 \in P$ and $m_0 \in M$.

Proposition A.0.14 *Let (P, p_0) be a principal G-bundle on (M, m_0), equipped with a flat connection A. Let $\rho^0 = \rho^0_A$ denote the holonomy representation of A, computed at the basepoints p_0 and m_0.*

If $\rho^0_g : \pi_1(M, m_0) \to G$ denotes the holonomy representation of A computed at the basepoints $p_0 \cdot g$ and m_0, then $\rho^0_g = g^{-1}\rho^0 g$.

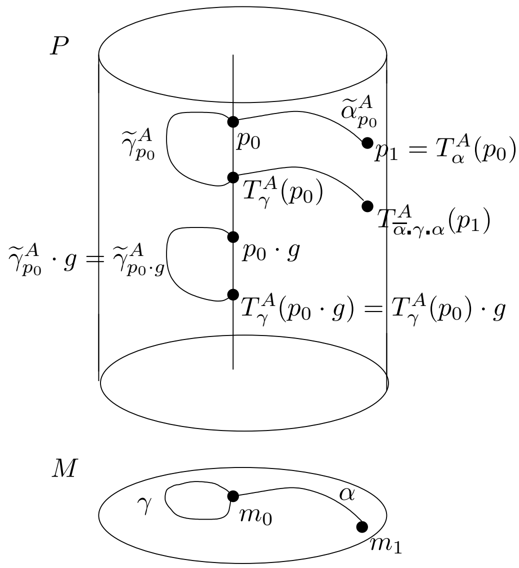

Figure A.1: Basepoints and holonomy.

Let α be a smooth path in M with $\alpha(0) = m_0$ and let $m_1 = \alpha(1)$. If $\rho^1 : \pi_1(M, m_1)$ denotes the holonomy representation of A computed at the basepoints $m_1 \in M$ and $p_1 = T^A_\alpha(p_0) \in P_{m_1}$, then ρ^1 and ρ^0 are identified under the isomorphism $\pi_1(M, m_0) \xrightarrow{\cong} \pi_1(M, m_1)$ induced by α. In other words, $\rho^1([\overline{\alpha}\centerdot\gamma\centerdot\alpha]) = \rho^0([\gamma])$ for any $[\gamma] \in \pi_1(M, m_0)$.

Proof. For the first statement, we need to show that for any loop γ based at $m_0 \in M$,

$$p_0 \cdot g = T^A_\gamma(p_0 \cdot g) \cdot g^{-1}\rho([\gamma])g.$$

By G-equivariance of parallel transport (Lemma A.0.10), the right-hand side is just

$$T^A_\gamma(p_0) \cdot \rho([\gamma])g = p_0 \cdot g,$$

as desired.

For the second statement, we need to show that

$$T^A_{\overline{\alpha}\centerdot\gamma\centerdot\alpha}(p_1) \cdot \rho^0([\gamma]) = p_1.$$

We have:

$$\begin{aligned} T^A_{\overline{\alpha}\bullet\gamma\bullet\alpha}(p_1)\cdot\rho^0([\gamma]) &= \left(T^A_\alpha T^A_\gamma T^A_{\overline{\alpha}}(T^A_\alpha(p_0)\right)\cdot\rho^0([\gamma]) \\ &= T^A_\alpha\left(T^A_\gamma(p_0)\right)\cdot\rho^0([\gamma]) \\ &= T^A_\alpha(p_0) \\ &= p_1. \end{aligned}$$

□

We denote the set of all (smooth) flat connections on a principal bundle P by $\mathcal{A}_{\text{flat}}(P)$. We now want to study the effect of the gauge group on holonomy.

Proposition A.0.15 *For any $A \in \mathcal{A}_{\text{flat}}(P)$ and any $\phi \in \mathcal{G}(P)$ we have*

$$\rho_{\phi_* A} = \phi_{m_0}\rho_A\phi_{m_0}^{-1},$$

where $\phi_{m_0} \in G$ is the unique element such that $p_0 \cdot \phi_{m_0} = \phi(p_0)$. (Note that with this definition, the map $\phi \mapsto \phi_{m_0}$ is a homomorphism $\mathcal{G}(P) \to G$.)

Proof. We must show that for any smooth curve $\gamma : [0,1] \to M$,

$$T_\gamma^{\phi_* A}(p_0)\cdot(\phi_{m_0}\rho_A([\gamma])\phi_{m_0}^{-1}) = p_0. \tag{A.1}$$

Let $\tilde{\gamma}^A_{p_0}$ denote the A-horizontal lift of γ starting at p_0. We claim that $\tilde{\gamma}^{\phi_* A}_{p_0} = (\phi\circ\tilde{\gamma}^A_{p_0})\cdot\phi_{m_0}^{-1}$. It suffices (by Lemma A.0.10 a)) to check that $\phi\circ\tilde{\gamma}^A_{p_0}$ is $\phi_* A$-horizontal. This is just a calculation:

$$(\phi\circ\tilde{\gamma}^A_{p_0})'(t) = D\phi\left((\tilde{\gamma}^A_{p_0})'(t)\right) = D\phi\left(A(\gamma'(t),\tilde{\gamma}^A_{p_0}(t))\right)$$

and

$$\begin{aligned} (\phi_* A)\left(\gamma'(t),\phi(\tilde{\gamma}^A_{p_0}(t))\right) &= D\phi\circ A\circ\tilde{\phi}^{-1}\left(\gamma'(t),\phi(\tilde{\gamma}^A_{p_0}(t))\right) \\ &= D\phi\circ A(\gamma'(t),\tilde{\gamma}^A_{p_0}(t)) \end{aligned}$$

as well. So $T_\gamma^{\phi_* A}(p_0) = \phi(T_\gamma^A(p_0)) \cdot \phi_{m_0}^{-1}$, and the left and side of (A.1) is now

$$\begin{aligned} T_\gamma^{\phi_* A}(p_0) \cdot (\phi_{m_0} \rho_A([\gamma]) \phi_{m_0}^{-1}) &= \phi(T_\gamma^A(p_0)) \cdot \phi_{m_0}^{-1} \cdot (\phi_{m_0} \rho_A([\gamma]) \phi_{m_0}^{-1}) \\ &= \phi(T_\gamma^A(p_0) \cdot \rho_A([\gamma])) \cdot \phi_{m_0}^{-1} = \phi(p_0) \cdot \phi_{m_0}^{-1} = p_0. \end{aligned}$$

□

Definition A.0.16 *The based gauge group $\mathcal{G}_0(P) \subset \mathcal{G}(P)$ is the kernel of the restriction homomorphism $r : \mathcal{G}(P) \to G$, $r(\phi) = \phi_{m_0}$. Equivalently, $\mathcal{G}_0(P)$ is the subgroup of gauge transformations which are the identity on P_{m_0}.*

An immediate consequence of Proposition A.0.15 is:

Corollary A.0.17 *The based gauge group $\mathcal{G}_0(P)$ acts trivially on holonomy; that is, for all $\phi \in \mathcal{G}_0(P)$ and all $A \in \mathcal{A}_{\text{flat}}(P)$ we have*

$$\rho_{\phi_* A} = \rho_A.$$

Holonomy defines a map

$$\mathcal{H} : \coprod_{[P]} \mathcal{A}_{\text{flat}}(P) \to \text{Hom}(\pi_1(M, m), G) \tag{A.2}$$

via the formula $\mathcal{H}(A) = \rho_A$. The disjoint union ranges over some chosen set of representatives for the *unbased* isomorphism classes of (based) principal G-bundles. In other words, we choose a set of representatives for the unbased isomorphism classes, and then choose, arbitrarily, a basepoint in each representative (at which we compute holonomy).

The set $\mathcal{A}_{\text{flat}}(P)$ will often be empty; in fact we will see that if M is compact (or more generally, if $\pi_1 M$ is finitely generated) and G is either compact or a real algebraic variety, then only finitely many isomorphism types of principal G-bundles on M admit flat connections.

Corollary A.0.17 shows that we have a diagram

$$\begin{tikzcd}
\coprod_{[P]} \mathcal{A}_{\text{flat}}(P) \arrow[rr, "\mathcal{H}"] \arrow[dr] & & \operatorname{Hom}(\pi_1(M, m_0), G) \\
& \coprod_{[P]} \mathcal{A}_{\text{flat}}(P)/\mathcal{G}_0(P) \arrow[ur, "\overline{\mathcal{H}}"] &
\end{tikzcd}$$

Our next goal is to explain the equivariance properties of this diagram. Proposition A.0.15 is a sort of equivariance statement for $\mathcal{H}$, at least when restricted to a particular bundle P. To understand equivariance for the map $\overline{\mathcal{H}}$, we need a lemma regarding the existence of gauge transformations with prescribed values.

Lemma A.0.18 *Assume G is connected. Then for any principal G-bundle (P, p_0) on M, and any $g \in G$, there exists a gauge transformation $\phi^g \in \mathcal{G}(P)$ such that $(\phi^g)_{m_0} = g$, i.e. $\phi_g(p_0) = p_0 \cdot g$. Hence the restriction homomorphism $\mathcal{G}(P) \to G$, $\phi \mapsto \phi_{m_0}$, is surjective.*

Remark A.0.19 *The argument below shows that for* any *group G, the image of the restriction map is a union of connected components of G. In general, though, I do not know how to prove that this map is surjective.*

Proof.[Proof of Lemma A.0.18] Consider the adjoint bundle (see [6, Section 2])

$$\operatorname{Ad} P = (P \times G)/G$$

where G acts via $(p, g) \cdot h = (p \cdot h, h^{-1}gh)$. The projection $\operatorname{Ad} P \to M$, $(p, g) \mapsto \pi(p)$, makes $\operatorname{Ad} P$ a (locally trivial) fiber bundle with fiber G (although it is *not* a principal G-bundle) and there is a one-to-one correspondence between sections of $\operatorname{Ad} P$ and gauge transformations of P, given by sending a section $s : M \to \operatorname{Ad} P$ to the gauge transformation $\phi^s : P \to P$ given by

$$\phi^s(p) = p \cdot g_s(p),$$

where $g_s(p) \in G$ is the unique element satisfying $s(\pi(p)) = [p, g_s(p)] \in \operatorname{Ad} P$. (See [6] for more details.) It now suffices to show that for each $g \in G$ there exists a section s of $\operatorname{Ad} P$ with $g_s(p_0) = g$. Note that $\operatorname{Ad} P$ has a canonical section $e(m) = [p, e]$

($p \in P_m$ arbitrary), corresponding to the identity element in $\mathcal{G}(P)$, and $\operatorname{Ad} P$ is trivial on some neighborhood of $m_0 \in M$. Since G is connected, we may modify the section $e(m)$ in this neighborhood in order to make it take any given value over the basepoint. $\square$

When G is connected, we now see that G acts on the space

$$\coprod_{[P]} \mathcal{A}_{\text{flat}}(P)/\mathcal{G}_0(P).$$

Specifically, the action of $g \in G$ on an equivalence class $[A] \in \mathcal{A}_{\text{flat}}(P)/\mathcal{G}_0(P)$ is given by the formula

$$g \cdot [A] = [(\phi^g)_* A],$$

where $\phi^g \in \mathcal{G}(P)$ is any gauge transformation satisfying $(\phi^g)_{m_0} = g$ (the existence of ϕ^g is guaranteed by Lemma A.0.18). This action is well-defined by Corollary A.0.17.

We can now state the main result of this appendix.

Theorem A.0.20 *The holonomy map defines a (continuous) bijection*

$$\overline{\mathcal{H}} : \coprod_{[P]} \mathcal{A}_{\text{flat}}(P)/\mathcal{G}_0(P) \to \operatorname{Hom}(\pi_1 M, G),$$

and if G is connected then this map is G-equivariant with respect to the G-action described above.

We begin by noting that equivariance is immediate from Proposition A.0.15, and continuity of the holonomy map is immediate from its definition in terms of integral curves of vector fields (here we are thinking of the C^∞-topology on $\mathcal{A}_{\text{flat}}(P)$).

In order to prove bijectivity of $\overline{\mathcal{H}}$, we will need to introduce the mixed bundles associated to representations $\rho : \pi_1 M \to G$.

Definition A.0.21 *Let $\rho : \pi_1 M \to G$ be a representation. We define the* mixed bundle *$E_\rho = \widetilde{M} \times_\rho G$ by*

$$E_\rho = \left(\widetilde{M} \times G\right) \Big/ (x, g) \sim (x \cdot \gamma, \rho(\gamma)^{-1} g)$$

Here $\widetilde{M} \xrightarrow{\pi_{\widetilde{M}}} M$ is the universal cover of M, considered as a principal $\pi_1 M$-bundle and equipped with a basepoint $\widetilde{m}_0 \in \widetilde{M}$ lying over $m_0 \in M$. The equivalence relation is defined for all $(x, g) \in \widetilde{M} \times G$ and all $\gamma \in \pi_1 M$.

It is easy to check that E_ρ is a principal G-bundle on M, with projection $[(\widetilde{m}, g)] \mapsto \pi_{\widetilde{M}}(\widetilde{m})$. We denote this map by $\pi_\rho : E_\rho \to M$. Note that since we have chosen basepoints $m_0 \in M$ and $\widetilde{m}_0 \in \widetilde{M}$, E_ρ acquires a canonical basepoint $[(\widetilde{m}_0, e)] \in E_\rho$ making E_ρ a based principal G-bundle. (Here $e \in G$ denotes the identity element.)

In fact, E_ρ also admits a canonical flat connection $\mathbb{A}_\rho$. We must define a G-equivariant splitting of the natural map

$$T(E_\rho) \xrightarrow{\alpha} \pi_\rho^* TM.$$

This is given by the formula

$$\mathbb{A}_\rho \left([\tilde{x}, g], \vec{\mathbf{v}}_x \right) = Dq \left(\left(D_{\tilde{x}} \pi_{\widetilde{M}} \right)^{-1} (\vec{\mathbf{v}}_x), \vec{\mathbf{0}}_g \right).$$

On the left, $x \in M$, $\vec{\mathbf{v}}_x \in T_x M$, $\tilde{x} \in \pi_{\widetilde{M}}^{-1}(x) \subset \widetilde{M}$, and $g \in G$. On the right, $\vec{\mathbf{0}}_g \in T_g G$ denotes the zero vector, q denotes the quotient map $\widetilde{M} \times G \to \widetilde{M} \times_\rho G = E_\rho$, and $D_{\tilde{x}} \pi_{\widetilde{M}}$ is invertible because $\pi_{\widetilde{M}} : \widetilde{M} \to M$ is a covering map.

Lemma A.0.22 *The map $\mathbb{A}_\rho$ is a well-defined, G-equivariant splitting of the natural map $\alpha : TE_\rho \to \pi_\rho^* TM$; in other words, $\mathbb{A}_\rho$ is a connection on E_ρ.*

Proof. To show that $\mathbb{A}_\rho$ is well-defined, let $\gamma \in \pi_1 M$ be any element. We must check that

$$Dq \left(\left(D_{\tilde{x}} \pi_{\widetilde{M}} \right)^{-1} (\vec{\mathbf{v}}_x), \vec{\mathbf{0}}_g \right) = Dq \left(\left(D_{\tilde{x} \cdot \gamma} \pi_{\widetilde{M}} \right)^{-1} (\vec{\mathbf{v}}_x), \vec{\mathbf{0}}_{\rho(\gamma)^{-1} g} \right).$$

By abuse of notation, for each $\gamma \in \pi_1 M$ we let γ denote the maps $\widetilde{M} \to \widetilde{M}$, $\tilde{x} \mapsto \tilde{x} \cdot \gamma$ and $\widetilde{M} \times G \to \widetilde{M} \times G$, $(\tilde{x}, g) \mapsto (\tilde{x} \cdot \gamma, \rho(\gamma)^{-1} \cdot g)$, i.e. the maps defining the actions of $\pi_1 M$. Then since $\pi_{\widetilde{M}} \circ \gamma = \pi_{\widetilde{M}}$ we have

$$\left(D_{\tilde{x} \cdot \gamma} \pi_{\widetilde{M}} \right)^{-1} = D_{\tilde{x}} \gamma \circ \left(D_{\tilde{x}} \pi_{\widetilde{M}} \right)^{-1},$$

and since $q \circ \gamma = q$ we have $Dq\, D\gamma = Dq$. Thus

$$\begin{aligned} Dq\left((D_{\tilde{x}\cdot\gamma})^{-1}(\vec{\mathbf{v}}_x), \vec{\mathbf{0}}_{\rho(\gamma)^{-1}g}\right) &= Dq\left(D_{\tilde{x}}\gamma \circ \left(D_{\tilde{x}}\pi_{\widetilde{M}}\right)^{-1}(\vec{\mathbf{v}}_x), \vec{\mathbf{0}}_{\rho(\gamma)^{-1}g}\right) \\ &= Dq\, D\gamma\left(\left(D_{\tilde{x}}\pi_{\widetilde{M}}\right)^{-1}(\vec{\mathbf{v}}_x), \vec{\mathbf{0}}_g\right) \\ &= Dq\left(\left(D_{\tilde{x}}\pi_{\widetilde{M}}\right)^{-1}(\vec{\mathbf{v}}_x), \vec{\mathbf{0}}_g\right), \end{aligned}$$

as desired.

For equivariance, we must check that $\mathbb{A}_\rho\left([\tilde{x}, gh], \vec{\mathbf{v}}_x\right) = \mathbb{A}_\rho\left([\tilde{x}, g], \vec{\mathbf{v}}_x\right) \cdot h$, i.e.

$$Dq\left(\left(D_{\tilde{x}}\pi_{\widetilde{M}}\right)^{-1}(\vec{\mathbf{v}}_x), \vec{\mathbf{0}}_{gh}\right) = Dq\left(\left(D_{\tilde{x}}\pi_{\widetilde{M}}\right)^{-1}(\vec{\mathbf{v}}_x), \vec{\mathbf{0}}_g\right) \cdot h.$$

This formula is immediate from G-equivariance of q.

Finally, we must check that $\mathbb{A}_\rho$ is a splitting, i.e. that $\alpha \circ \mathbb{A}_\rho = \mathrm{Id}_{\pi_\rho^* TM}$. This follows from commutativity of the diagram

$$\begin{tikzcd} \widetilde{M} \times G \arrow[r, "q"] \arrow[rr, bend left, "\pi_{\widetilde{M}} \circ p_1"] & E_\rho \arrow[r, "\pi_\rho"] & M \end{tikzcd}.$$

Specifically,

$$\begin{aligned} \alpha\left(\mathbb{A}_\rho\left([\tilde{x}, g], \vec{\mathbf{v}}_x\right)\right) &= \alpha\left(Dq\left(\left(D_{\tilde{x}}\pi_{\widetilde{M}}\right)^{-1}(\vec{\mathbf{v}}_x), \vec{\mathbf{0}}_g\right)\right) \\ &= \left([\tilde{x}, g], D\pi_\rho\, Dq\left(\left(D_{\tilde{x}}\pi_{\widetilde{M}}\right)^{-1}(\vec{\mathbf{v}}_x), \vec{\mathbf{0}}_g\right)\right) \\ &= \left([\tilde{x}, g], D\left(\pi_{\widetilde{M}} \circ p_1\right)\left(\left(D_{\tilde{x}}\pi_{\widetilde{M}}\right)^{-1}(\vec{\mathbf{v}}_x), \vec{\mathbf{0}}_g\right)\right) \\ &= \left([\tilde{x}, g], \vec{\mathbf{v}}_x\right) \end{aligned}$$

as desired. □

Proposition A.0.23 *The connection $\mathbb{A}_\rho$ is flat, with holonomy representation*

$$\mathcal{H}(\mathbb{A}_\rho) = \rho.$$

Proof. To show that $\mathbb{A}_\rho$ is flat, it suffices (by Proposition A.0.11) to check local triviality. Let $U \subset M$ be an open set over which $\widetilde{M}$ is trivial. Over this neighborhood, E_ρ is simply

$$(U \times \pi_1 M \times G)/\pi_1 M \cong U \times G,$$

and $\mathbb{A}_\rho$ is clearly identified with the trivial connection on $U \times G$ under this isomorphism.

To compute $\mathcal{H}(\mathbb{A}_\rho)$, let $\gamma : I \to M$ be a smooth loop based at $m_0 \in M$. We must show that if $\tilde{\gamma}_\rho$ is the $\mathbb{A}_\rho$-horizontal lift of γ with $\tilde{\gamma}_\rho(0) = [\widetilde{m}_0, e]$, then

$$\tilde{\gamma}_\rho(1) = [\widetilde{m}, e] \cdot \rho(\gamma) = [\widetilde{m}, \rho(\gamma)].$$

To begin, let $\tilde{\gamma}_{\widetilde{M}} : I \to \widetilde{M}$ be the unique lift of γ to $\widetilde{M}$ with $\tilde{\gamma}_{\widetilde{M}}(0) = \widetilde{m}_0$. Then by definition of the principal bundle structure on $\widetilde{M}$, we have $\tilde{\gamma}_{\widetilde{M}}(1) = \widetilde{m} \cdot [\gamma]$. The $\mathbb{A}_\rho$-horizontal lift of γ to $E_\rho = \left(\widetilde{M} \times G\right) \Big/ \pi_1 M$ is now given by

$$\tilde{\gamma}_\rho(t) = \left[\tilde{\gamma}_{\widetilde{M}}(t), e\right] = q\left(\tilde{\gamma}_{\widetilde{M}}(t), e\right),$$

where $e \in G$ denotes the identity element. Indeed, $\tilde{\gamma}_\rho(0) = [\tilde{\gamma}_{\widetilde{m}}(0), e] = [\widetilde{m}_0, e]$, and

$$\begin{aligned} \tilde{\gamma}_\rho'(t) &= Dq\left(\tilde{\gamma}_{\widetilde{M}}'(t), \vec{\mathbf{0}}_e\right) = Dq\left(\left(D_{\tilde{\gamma}_{\widetilde{M}}(t)}\pi_{\widetilde{M}}\right)^{-1}(\gamma'(t)), \vec{\mathbf{0}}_e\right) \\ &= \mathbb{A}_\rho\left(\tilde{\gamma}_\rho(t), \gamma'(t)\right) \end{aligned}$$

as desired. We now have

$$\tilde{\gamma}_\rho(1) = \left[\tilde{\gamma}_{\widetilde{M}}(1), e\right] = [\widetilde{m}_0 \cdot \gamma, e] = [\widetilde{m}_0, \rho(\gamma)e] = [\widetilde{m}_0, \rho(\gamma)],$$

completing the proof. □

Complementing this result, we have:

Proposition A.0.24 *Let (P, p_0) be a based principal G-bundle over M, equipped with a flat connection $A \in \mathcal{A}_{\text{flat}}(P)$. Then there is an isomorphism of based principal G-bundles*

$$\phi : (P, p_0) \xrightarrow{\cong} \left(E_{\mathcal{H}(A)}, [\widetilde{m}_0, e]\right)$$

such that $\phi_* A = \mathbb{A}_{\mathcal{H}(A)}$.

In light of Proposition A.0.23, we see that Proposition A.0.24 is a special case of the following result.

Proposition A.0.25 *Let* (P, p_0) *and* (Q, q_0) *be based principal* G*-bundles over* M *with flat connections* A *and* B*, respectively. If* $\mathcal{H}(A) = \mathcal{H}(B)$*, then there is a based isomorphism* $\phi : P \to Q$ *such that* $\phi_* A = B$.

Proof. We define ϕ by defining its restrictions $\phi_m := \phi|_{P_m}$ to each fiber of P, beginning with the fiber P_{m_0} over $m_0 \in M$. Since we require $\phi(p_0) = q_0$, equivariance forces us to define $\phi(p_0 \cdot g) = q_0 \cdot g$ for any other point $p_0 \cdot g \in P_{m_0}$. Now, given another point $m \in M$, choose a smooth path $\gamma : I \to M$ with $\gamma(0) = m_0$ and $\gamma(1) = m$. Then we define ϕ_m on P_m via parallel transport:

$$\phi_m = T^B_\gamma \circ \phi_{m_0} \circ T^A_{\overline{\gamma}}.$$

Since each of the maps on the right is G-equivariant, so is their composition ϕ_m.

We must check that our definition of ϕ_m is independent of the chosen path γ. It is easy to check (using Lemma A.0.10) that this will follow once we show that for every smooth loop $\gamma : I \to M$ based at m_0, we have

$$T^B_\gamma \circ \phi_{m_0} \circ T^A_{\overline{\gamma}} = \phi_{m_0}. \tag{A.3}$$

To check (A.3), we begin by noting that $T^A_{\overline{\gamma}}(p_0) \cdot \rho_A([\overline{\gamma}]) = p_0$, so we have

$$\begin{aligned} T^B_\gamma \circ \phi_{m_0} \circ T^A_{\overline{\gamma}}(p_0) &= \left(T^B_\gamma \circ \phi_{m_0}\right)(p_0 \cdot \rho_A([\gamma])) \\ &= T^B_\gamma \left(q_0 \cdot \rho_A([\gamma])\right) = T^B_\gamma(q_0) \cdot \rho_A([\gamma]). \end{aligned}$$

But by assumption, A and B have the same holonomy, so $T^B_\gamma(q_0) \cdot \rho_A([\gamma]) = q_0$. Thus we have

$$T^B_\gamma \circ \phi_{m_0} \circ T^A_{\overline{\gamma}}(p_0) = q_0 = \phi_{m_0}(p_0),$$

and (A.3) follows by equivariance. Hence the map ϕ is well-defined. We note that ϕ is easily seen to be smooth: the connections A and B are locally trivial by Proposition A.0.11, and parallel transport for the trivial connection is clearly smooth.

To prove that $\phi_* A_P = A_Q$, we work locally. Given $p \in P$, choose a path $\gamma : [0,1] \to M$ with $\gamma(0) = m_0$, $\gamma(1) = \pi_P(p)$ and cover $\gamma([0,1]) \subset M$ by open sets $U_1, ..., U_k$ over which the connections A_P and A_Q are both trivial. We may now subdivide γ into subpaths $\gamma_i : [t_{i-1}, t_i] \to M$, where $i = 1, ..., k$, $t_0 = 0, t_k = 1$ and (after renumbering the U_i if necessary) $\gamma_i([t_{i-1}, t_i]) \subset U_i$. Since A_P and A_Q are both trivial over U_i, we may choose isomorphisms

$$\psi_i : P|_{U_i} \to Q|_{U_i}$$

such that $(\psi_i)_* A_P = A_Q$. Moreover, we may choose the ψ_i in order and assume that $\psi_1(p_0) = q_0$, and then (inductively) we may assume that $\psi_i = \psi_{i+1}$ on the fiber over t_i (here we use the fact that, for any $h \in G$, the trivial connection on $U_i \times G$ is fixed by the constant gauge transformation $(u, g) \mapsto (u, hg)$). It will now suffice to check that $\psi_i = \phi|_{U_i}$. This is easily proven by induction on k, using the following lemma.

Lemma A.0.26 *Let $\phi : (P, p_0) \to (Q, q_0)$ be a map of principal G-bundles, and let A be a flat connection on P. Then for any $p \in P$, $\phi(p)$ is given by the formula*

$$\phi(p) = T_\gamma^{\phi_* A} \circ \phi_{m_0} \circ T_{\overline{\gamma}}^A(p),$$

where $\gamma : [0,1] \to M$ is any smooth path with $\gamma(0) = m_0$ and $\gamma(1) = \pi_P(p)$, and ϕ_{m_0} is the restriction of ϕ to the fiber over m_0 (so $\phi_{m_0}(p_0 \cdot g) = q_0 \cdot g$).

Proof. If $\tilde{\gamma}_A$ denotes the A-horizontal lift of γ, with $\tilde{\gamma}_A(1) = p$, then $\phi \circ \tilde{\gamma}_A = \tilde{\gamma}_{\phi_* A}$ is the $(\phi_* A)$-horizontal lift of γ ending at $\phi(p)$ (cf. the proof of Proposition A.0.15). So we have

$$\begin{aligned} T_\gamma^{\phi_* A} \circ \phi_{m_0} \circ T_{\overline{\gamma}}^A(p) &= T_\gamma^{\phi_* A}(\phi \circ \tilde{\gamma}_A(0)) \\ &= T_\gamma^{\phi_* A}(\tilde{\gamma}_{\phi_* A}(0)) = \tilde{\gamma}_{\phi_* A}(1) = \phi(p). \end{aligned}$$

□

This complete the proof of Proposition A.0.25. We can now bring together our results to prove Theorem A.0.20.

Proof of Theorem A.0.20. We just need to check that the holonomy map

$$\overline{\mathcal{H}} : \coprod_{[P]} \mathcal{A}_{\text{flat}}(P)/\mathcal{G}_0(P) \to \text{Hom}(\pi_1 M, G)$$

is both surjective and injective. Surjectivity follows immediately from Proposition A.0.23, which produces, for any $\rho \in \text{Hom}(\pi_1 M, G)$, a bundle E_ρ with a connection $\mathbb{A}_\rho$ such that $\mathcal{H}(\mathbb{A}_\rho) = \rho$.

For injectivity, say $\mathcal{H}(A) = \mathcal{H}(B)$, where $A \in \mathcal{A}_{\text{flat}}(P)$ and $B \in \mathcal{A}_{\text{flat}}(Q)$. Then by Proposition A.0.25, we have an isomorphism $\phi : P \xrightarrow{\cong} Q$ such that $\phi_* A = B$. But if P and Q came from our chosen set of representatives for the isomorphism classes of principal G-bundles, then we must have $P = Q$, and moreover ϕ is now an element of the based gauge group $\mathcal{G}_0(P)$. So $[A] = [B] \in \coprod_{[P]} \mathcal{A}_{\text{flat}}(P)/\mathcal{G}_0(P)$, proving injectivity. □

As an easy consequence of this result, we obtain the following more well-known statement. (A proof of this corollary is given by Morita [36, Theorem 2.9]. However, Morita does not prove an analogue of Proposition A.0.25 and consequently his argument does not make the injectivity portion clear.)

Corollary A.0.27 *Let G be a connected Lie group. Then there is a bijection between isomorphism classes of flat principal G-bundles over M and conjugacy classes of representations $\rho : \pi_1 M \to G$. In other words, holonomy defines a (continuous) bijection*

$$\coprod_{[P]} \mathcal{A}_{\text{flat}}(P)/\mathcal{G}(P) \xrightarrow{\cong} \text{Hom}(\pi_1 M, G)/G.$$

Proof. Surjectivity is immediate from Theorem A.0.20. For injectivity, say $\mathcal{H}(A) = g\mathcal{H}(B)g^{-1}$ for some $g \in G$, where $A \in \mathcal{A}_{\text{flat}}(P)$, $B \in \mathcal{A}_{\text{flat}}(Q)$. Let $p_0 \in P$ and $q_0 \in Q$ be the given basepoints. Then by Proposition A.0.14, we know that the connections A and B have the same holonomy, if we compute holonomy of B at $q_0 \cdot g$. Hence Proposition A.0.25 gives an isomorphism

$$\phi : (P, p_0) \longrightarrow (Q, q_0 \cdot g)$$

such that $\phi_* A = B$. But existence of this isomorphism implies that $P = Q$ (and $p_0 = q_0$), and now ϕ is an unbased gauge transformation of P. $\square$

The following corollary is the version of Theorem A.0.20 stated in [14], and identifies those representations arising from a given bundle P.

Corollary A.0.28 *Let (P, p_0) be a based principal G-bundle on M, and let*

$$\mathrm{Hom}_P(\pi_1(M, m_0), G) = \{\rho \ : \ (E_\rho, [\widetilde{m}_0, e]) \cong (P, p_0)\}.$$

Then holonomy induces a continuous bijection

$$\mathcal{A}_{\mathrm{flat}}(P)/\mathcal{G}_0(P) \xrightarrow{\cong} \mathrm{Hom}_P(\pi_1(M, m_0), G).$$

Proof. By Theorem A.0.20, holonomy induces a continuous injection

$$\mathcal{A}_{\mathrm{flat}}(P)/\mathcal{G}_0(P) \overset{\overline{\mathcal{H}}}{\hookrightarrow} \mathrm{Hom}(\pi_1(M, m_0), G),$$

so we just need to identify the image. If $A \in \mathcal{A}_{\mathrm{flat}}(P)$, then $(E_{\mathcal{H}(A)}, [\widetilde{m}_0, e]) \cong (P, p_0)$ by Proposition A.0.24, so $\overline{\mathcal{H}}([A]) \in \mathrm{Hom}_P(\pi_1(M, m_0), G)$. Conversely, if $\rho \in \mathrm{Hom}_P(\pi_1(M, m_0), G)$ then we are given an isomorphism

$$\phi : (E_\rho, [\widetilde{m}_0, e]) \xrightarrow{\cong} (P, p_0);$$

by Proposition A.0.23 we know that the connection $\mathbb{A}_\rho$ on E_ρ has holonomy ρ (at $[\widetilde{m}_0, e]$), and hence the connection $\phi_* \mathbb{A}_\rho$ on P has holonomy ρ (at $\phi([\widetilde{m}_0, e]) = p_0$). $\square$

Remark A.0.29 *We note that the space $\mathrm{Hom}_P(\pi_1 M, G)$ is always a union of connected components in $\mathrm{Hom}(\pi_1 M, G)$. This is easily shown using the method in the proof of Proposition A.0.30 below.*

Using the techniques of this appendix, we can now prove the finiteness result needed in Section 4.2.

Proposition A.0.30 *Assume that $\pi_1 M$ is finitely generated, and that G is a real algebraic variety (i.e. a subset of $\mathbb{R}^n$ cut out by polynomial equations). Then only finitely many isomorphism classes of principal G-bundles over M admit flat connections.*

Proof. We will apply a theorem of Whitney [49], stating that any real algebraic variety has finitely many path components. (To be precise, Whitney proves that real algebraic varieties have finitely many *topological* components, but such varieties are also triangulable [21] and hence their topological and path components coincide.) Now, since $\pi_1 M$ is finitely generated, we know that $\mathrm{Hom}(\pi_1 M, G)$ embeds into G^k for some k, and in fact the representation space is precisely the subvariety of G^k cut out by the relations in $\pi_1 M$. (Note here that even if $\pi_1 M$ is not finitely presented, the ideal defined by the relations in $\pi_1 M$ will be finitely generated, by the Hilbert Basis Theorem.) Thus we conclude that $\mathrm{Hom}(\pi_1 M, G)$ has finitely many path components.

Now, let $\rho_1, ..., \rho_n \in \mathrm{Hom}(\pi_1 M, G)$ denote a set of representatives for the path components of $\mathrm{Hom}(\pi_1 M, G)$, and let $P \to M$ denote a principal G-bundle admitting a flat connection A. We will show that $P \cong E_{\rho_i}$ for some $i \in \{1, ..., n\}$, where E_{ρ_i} denotes the mixed bundle from Definition A.0.21. First, we know from Proposition A.0.24 that $P \cong E_{\mathcal{H}(A)}$. Now, choose i such that $\mathcal{H}(A)$ and ρ_i lie in the same path component of $\mathrm{Hom}(\pi_1 M, G)$. Then there exists a path $R : [0,1] \to \mathrm{Hom}(\pi_1 M, G)$ such that $R_0 = \mathcal{H}(A)$ and $R_1 = \rho_i$. Consider the bundle

$$E_R = (\widetilde{M} \times [0,1] \times G) \Big/ (\widetilde{m}, t, g) \sim (\widetilde{m} \cdot \gamma, t, R_t(\gamma)^{-1} g).$$

This is a principal G-bundle over $M \times [0,1]$, with projection

$$[(\widetilde{m}, t, g)] \mapsto (\pi_{\widetilde{M}}(\widetilde{m}), t);$$

note that if $U \subset M$ is an open set over which $\widetilde{M}$ is trivial, then E_R is trivial over $U \times [0,1]$. The Bundle Homotopy Theorem now gives us an isomorphism $E_R|_{M \times \{0\}} \cong E_R|_{M \times \{1\}}$, and clearly these bundles are just $E_{\mathcal{H}(A)}$ and E_{ρ_i}, respectively. $\square$

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