Math 204-2, Fall 2008
Proof-Writing Assignment 2
DUE: Tuesday, 12/2/08

Please hand in solutions to both problems. Your answers will be graded on both correctness and on clarity of presentation. Please make sure to write clearly, and think carefully about the logic of your arguments. Remember that a proof is a logical sequence of deductions, and you should make clear why each step in your proof follows from the previous one.

A) The purpose of this problem is to think carefully about two fundamental results regarding vector spaces. We've already seen these results in class, on the homework, and on exams. Please give a clear and complete proof of each of the following results (if you use a result from a previous homework or test, include a proof of that result).

i) Prove that if $\{v_1, \ldots, v_k\}$ is a *linearly independent* subset of a finite dimensional vector space V, then there is some collection of vectors $\{v_{k+1}, \ldots, v_n\}$ such that $\{v_1, \ldots, v_n\}$ forms a basis for V.

ii) Prove that if V is a vector space containing a finite spanning set, then V is in fact finite dimensional.

B) Let U, V, and W be finite dimensional vector spaces, and let $S : U \to V$ and $T : V \to W$ be linear transformations. Recall that the composite transformation $T \circ S : U \to W$ is defined by $T \circ S(u) = T(S(u))$.

i) Prove that $\ker(S)$ is a subspace of $\ker(T \circ S)$, while $\text{range}(T \circ S)$ is a subspace of $\text{range}(T)$.

ii) Consider the case where $U = V = W$ and $S = T$. Prove that if $\text{range}(T \circ T) = \text{range}(T)$, then $\ker(T) \cap \text{range}(T) = \{0\}$. (Hint: first show, using part i) and the Rank-Nullity Theorem, that $\ker(T) = \ker(T \circ T)$. Now see what you can say about a vector v that lies in both $\ker(T)$ and in $\text{range}(T)$.)

C) Early in this course, we learned that row reduction does not change the solution set of a linear system. In other words, if two linear systems are row equivalent, then they have the same solution set. We now have the necessary tools to prove the converse of this statement. This is a great application of all the abstract things we've learned to a nice, concrete problem.

Theorem 1 Let $Ax = 0$ and $A'x = 0$ be two homogeneous $m \times n$ linear systems. If the solution sets for these linear systems are the same (i.e. if $\{x \in \mathbb{R}^n \mid Ax = 0\} = \{x \in \mathbb{R}^n \mid A'x = 0\}$) then there is some sequence of row operations that transforms the matrix A into the matrix A'.

The goal of this problem is to give a proof of this theorem. I suggest proving it by filling in the following outline.

i) Show that Theorem 1 is equivalent to the following result.

Theorem 2 If A and A' are $m \times n$ matrices with the same nullspace, then there exists an invertible matrix B such that $BA = A'$.

Now we'll prove Theorem 2.

ii) Let $\{v_1, \ldots, v_k\}$ be a basis for the nullspace of A and A'. Use problem A) part i) to obtain a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for $\mathbb{R}^n$.

iii) Show that the set $\{Av_{k+1}, \ldots, Av_n\}$ is linearly independent in $\mathbb{R}^m$. (Hint: this is very similar to the proof Theorem 5 in Section 5.4.)

iv) Show that there exists an *isomorphism* $T : \mathbb{R}^m \to R^m$ such that $T(Av_{k+1}) = A'v_{k+1}, T(Av_{k+2}) = A'v_{k+2}, \ldots, T(Av_n) = A'v_n$. (Hint: remember that you can define a linear transformation just by specifying its values on a basis. You'll have to use Problem A) part i) again.)

v) Now put the pieces together: show that if B is the matrix for your transformation T with respect to the standard basis for $\mathbb{R}^m$, then B is invertible and $BA = A'$.

Extra Credit: What can you say about the case of non-homogeneous systems? Is it still true that if $Ax = b$ and $A'x = b'$ have the same solution set, then the augmented system $(A|b)$ can be transformed into the augmented system $(A'|b')$ using row operations? (Hint: consider the relationship between solutions to the augmented system and solutions to the associated homogeneous system.)