THE HOMOTOPY LIMIT PROBLEM IN STABLE REPRESENTATION THEORY AND THE GEOMETRY OF FLAT CONNECTIONS (DRAFT)

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ABSTRACT. For a discrete group G, the relationship between Carlsson's deformation K–theory spectrum and the complex connective K–theory of BG can be viewed as a homotopy limit problem in the sense of Thomason, providing a natural map of $\mathbf{ku}$–algebras $K^{\mathrm{def}}(G) \to F(BG_+, \mathbf{ku})$. We show that this map is closely related to the natural maps $\mathrm{Hom}(G, U(n)) \longrightarrow \mathrm{Map}(BG, BU(n))$, and that it agrees on homotopy groups with the topological Atiyah–Segal map introduced in [3]. Using results of T. Lawson, we study this map for products of surface groups and for crystallographic groups, and give applications to questions about families of flat bundles and spaces of flat connections.

1. INTRODUCTION

There is a close relationship between representation theory and K–theory, first observed by Atiyah in the context of finite groups [1]. Given a discrete group G and a representation $\rho : G \to U(n)$, one may form the "mixed" vector bundle $EG \times_G \mathbb{C}^n \to BG$, which represents a class in $K^0(BG)$. It is an exercise to check that this bundle (or rather its associated principle $U(n)$–bundle) is classified by the map $B\rho : BG \to BU(n)$, where we use the functorial simplicial model for these classifying spaces. In Baird–Ramras [3], this map was extended to a homomorphism

$$\alpha_G \colon K_n^{\mathrm{def}}(G) \longrightarrow K^{-n}(BG), \tag{1}$$

where $K_n^{\mathrm{def}}(G)$ is the Grothendieck group of (unbased homotopy classes of) S^n–families of unitary representations of G. This map, which we call the *topological Atiyah–Segal map*, is closely related to the classical Novikov Conjecture on the homotopy invariance of higher signatures: if $B\Gamma$ is homotopy equivalent to a finite CW complex and (1) is rationally surjective for $n >> 0$, then Ramras–Willett–Yu [19, Lemma 3.15 and Corollary 4.3] implies that the analytical assembly map is rationally injective (which implies the Novikov Conjecture for G).

If G is finitely generated, one has an isomorphism

$$K_n^{\mathrm{def}}(G) \cong \pi_n(K^{\mathrm{def}}(G),$$

where $K^{\mathrm{def}}(G)$ is Carlsson's deformation K–theory spectrum. This spectrum is built from the representation spaces $\mathrm{Hom}(G, U(n))$, and computations have shown that in certain cases, its homotopy groups are closely related to the K–theory of BG.

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In this article, we present a construction, due to Carlsson (private communication), which provides a natural map

$$K^{\mathrm{def}}(G) \longrightarrow F(BG_+, \mathbf{ku}) \tag{2}$$

in the derived category of **ku**–algebras. Throughout the paper, F will denote the *based* mapping spectrum, and BG_+ will denote BG with a disjoint base point. This map is obtained by viewing $K^{\mathrm{def}}(G)$ and $F(BG_+, \mathbf{ku})$ as the fixed points and homotopy fixed points (respectively) of the K–theory of a certain G–equivariant category. This formulation fits into the notion of *homotopy limit problems*, as expounded by Thomason [23]. Briefly, a homotopy limit problem asks whether, given a space or spectrum X with a G–action, the natural inclusion

$$X^G \to X^{hG}$$

of fixed points into the homotopy fixed is a weak equivalence. A large number of problems in homotopy theory fit into this framework: the work of Atiyah mentioned above, and subsequent work of Atiyah and Segal [2]; the Sullivan conjecture [22, 14]; Segal's Burnside ring conjecture [21, 5]; and the Quillen–Lichtenbaum Conjecture (see, for example, [20]).

This paper establishes relationships between the homotopy limit problem for deformation K–theory and the natural maps

$$B : \mathrm{Hom}(G, U(n))) \longrightarrow \mathrm{Map}_*(BG, BU(n)). \tag{3}$$

Here Map_* denotes the based mapping space. These maps were related to the topological Atiyah–Segal map α_G in [3], and combining these relationships leads to the conclusion that homotopy limit problem induces α_G on homotopy groups. **A full discussion of this point still needs to be added to the paper.** This shows in particular that the topological Atiyah–Segal map is a map of **ku**–algebras, which is not at all apparent from the explicit description in [3] in terms of flat families.

Combined with previous work of the author [17], this leads to the conclusion that if G is a product fundamental groups of aspherical surfaces, then the natural map (2) induces an isomorphism on homotopy in dimensions greater than the rational cohomological dimension of G, minus 2 (Theorem 5.1). (At the moment, the proof only works in the orientable case, although the computations in [17] suggest that it should go through in the non-orientable case as well.) Similar (but somewhat weaker) results are obtained for crystallographic groups in Section 7. (As discussed above, these results imply the Novikov Conjecture for the groups in question. Significantly simpler proofs were given in Ramras–Willett–Yu [19] using non-spherical families.)

In Sections 6 and 8 we consider consequences of these results for families of flat vector bundles and homotopy groups of spaces of flat connections, building on the results in Baird–Ramras [3, Sections 3 and 5].

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2. Deformation K–theory

We now recall the construction of deformation K–theory given in [15]. For any discrete group G, we construct the topological permutative category $\mathrm{R}(G)$ with object space

$$\mathrm{Ob}\,\mathrm{R}(G) = \coprod_{n=0}^{\infty} \mathrm{Hom}(G, U(n))$$

and morphism space

$$\mathrm{Mor}\,\mathrm{R}(G) = \coprod_{n=0}^{\infty} \mathrm{Hom}(G, U(n)) \times U(n),$$

where the pair $(\rho, A) \in \mathrm{Mor}\,\mathrm{R}(G)$ is viewed as an isomorphism $\rho \to A\rho A^{-1}$, and composition is determined by $(B, A\rho A^{-1}) \circ (A, \rho) = (BA, \rho)$. When $n = 0$, we set $U(0) = \mathrm{Hom}(G, U(0)) = \{0\}$. Block sum of unitary matrices gives this category a permutative structure, with $0 \in \mathrm{Hom}(G, U(0))$ as unit, and the deformation K–theory spectrum of G is defined to be the K–theory spectrum associated to $\mathrm{R}(G)$. For background on topological permutative categories and their associated K–theory spectra, see [12].

3. The homotopy limit construction

In this section, we explain Carlsson's construction of the natural map

$$K^{\mathrm{def}}(G) \longrightarrow F(BG_+, \mathbf{ku}).$$

We begin by defining a topological permutative category on which G acts by permutative functors, such that the fixed point category is precisely the category $\mathrm{R}(G)$ giving rise to deformation K–theory.

Definition 3.1. *The category $\widetilde{\mathcal{R}}(G)$ has object space*

$$\mathrm{Ob}\,\widetilde{\mathcal{R}}(G) = \coprod_{n} \mathrm{Hom}(G, U(n))$$

and morphism space

$$\mathrm{Mor}\,\mathrm{R}(G) = \coprod_{n} \mathrm{Hom}(G, U(n)) \times \mathrm{Hom}(G, U(n)) \times U(n).$$

The morphism (ρ, ψ, A) has domain ψ and codomain ρ, and composition is given by

$$(\phi, \rho, B) \circ (\rho, \psi, A) = (\phi, \psi, BA).$$

We think of $\widetilde{\mathcal{R}}(G)$ as the category of G–equivariant Hermitian inner product spaces, with linear isometries as morphisms.

Again, block sum of unitary matrices gives $\widetilde{\mathcal{R}}(G)$ the structure of a topological permutative category. Moreover, $\widetilde{\mathcal{R}}(G)$ admits a (right) action of G by continuous permutative functors: given $\gamma \in G$, we define a functor (roughly, conjugation by γ)

$$c_\gamma : \widetilde{\mathcal{R}}(G) \longrightarrow \widetilde{\mathcal{R}}(G)$$

by $c_\gamma(\rho) = \rho$, and $c_\gamma(\rho, \psi, A) = (\rho, \psi, \rho(\gamma)^{-1} A \psi(\gamma))$. We leave it to the reader to verify that these maps are (continuous) permutative functors, and that all together they define a right action of G on $\widetilde{\mathcal{R}}(G)$.

Given a category $\mathcal{D}$ and a (right) action of a group H on $\mathcal{D}$ by functors, the *fixed point category* $\mathcal{D}^H$ is the category with objects $(\mathrm{Ob}\,\mathcal{D})^H$ and morphisms $(\mathrm{Mor}\,\mathcal{D})^H$. Note that if $\phi \cdot h = \phi$ for some $\phi \in \mathrm{Mor}\,\mathcal{D}$, then both the domain and codomain of ϕ must be fixed by h, and moreover if $\phi, \psi \in (\mathrm{Mor}\,\mathcal{D})^H$ are composable, then since h acts via a functor, $(\phi \circ \psi) \cdot h = (\phi \cdot h) \circ (\psi \cdot h) = \phi \circ \psi$. Hence $\mathcal{D}^H$ is a subcategory of $\mathcal{D}$.

Lemma 3.2. *There is an isomorphism of topological permutative categories* $\mathrm{R}(G) \cong \widetilde{\mathcal{R}}(G)^G$*, and consequently there is an isomorphism of K–theory spectra* $K^{\mathrm{def}}(G) \cong (\mathbb{K}(\widetilde{\mathcal{R}}(G)))^G$.

Proof. The inclusion functor $\mathrm{R}(G) \to \widetilde{\mathcal{R}}(G)$, defined on morphisms and objects by

$$\rho \mapsto \rho, \ \ (\rho, A) \mapsto (A\rho A^{-1}, \rho, A),$$

is an isomorphism onto the subcategory $\widetilde{\mathcal{R}}(G)^G \subset \widetilde{\mathcal{R}}(G)$. Note that the inclusion functor is permutative, and that $\widetilde{\mathcal{R}}(G)^G$ is a sub-permutative category of $\mathrm{R}(G)$ (meaning that it is closed under the block sum operation).

It is elementary to check that fixed points commute with the construction of the K–theory spectrum of a permutative category (in fact, fixed points commute with May's the functor [12] from permutative categories to Γ– categories). □

Next, we show that there is a natural weak equivalence between the *homotopy fixed points* of the action of G on $\mathbb{K}(\widetilde{\mathcal{R}}(\Gamma))$ and the mapping spectrum $F(BG_+, \mathbf{ku})$. Here we interpret $\mathbf{ku}$ as the K–theory spectrum of the topological permutative category **Vect** of finite-dimensional Hermitian inner product spaces and linear isometries (to be precise, the objects in **Vect** consist of the discrete set of natural numbers, and the morphisms are $\coprod_n U(n)$) as in [10]. Equivalently, $\mathbf{ku}$ is the deformation K–theory spectrum of the trivial group. Given an Ω spectrum $\mathbf{S}$, and a group H, an action of H on $\mathbf{S}$ is an action of H on each of the (based) spaces $S_0, S_1, \ldots$ in the spectrum S, so that the binding maps $\Sigma S_i \to S_{i+1}$ are equivariant (where $\Sigma S_i = S^1 \wedge S_i$ with trivial H–action on the S^1 factor). We call a spectrum with an H–action an H–spectrum. The (naive) homotopy fixed point spectrum of an H–spectrum $\mathbf{S}$ is then the equivariant mapping spectrum

$$\mathbf{S}^{hH} := F^H(EH, \mathbf{S}),$$

formed from the sequence of spaces $\mathrm{Map}^H(EH, S_i)$. Note that this is still an Ω–spectrum, because the homeomorphism

$$\mathrm{Map}(EH, \Omega S_i) \cong \Omega \mathrm{Map}(EH, S_i)$$

restricts to a homeomorphism

$$\mathrm{Map}^H(EH, \Omega S_i) \cong \Omega \mathrm{Map}^H(EH, S_i).$$

The following result is standard.

Proposition 3.3. *Let $\mathbf{S}$ and $\mathbf{T}$ be H–spectra, and let $f : \mathbf{S} \to \mathbf{T}$ be an H–equivariant map which is also a weak equivalence of spectra (in the non-equivariant sense). Then the induced map $\mathbf{S}^{hH} \to \mathbf{T}^{hH}$ is a weak equivalence as well.*

This may be proven by noting that the homotopy fixed point spectrum is the homotopy limit, over the one-object category BH, of the functor $BH \to$ **Spectra** representing the action. Since homotopy limits are homotopy invariant in the appropriate sense, the proposition follows.

In our case, the K–theory spectrum of a permutative category with H–action is an Ω-spectrum with H–action, and hence we can define the homotopy fixed point spetrum in the above manner.

Lemma 3.4. *There is a weak homotopy equivalence*

$$F(BG_+, \mathbf{ku}) \xrightarrow{\simeq} \mathbb{K}(\widetilde{\mathcal{R}}(G))^{hG}$$

Proof. Consider the full subcategory $\widetilde{\mathcal{R}}_I(G) \subset \widetilde{\mathcal{R}}(G)$ on the trivial representations I_n, $n = 0, 1, 2, \ldots$; note that $\widetilde{\mathcal{R}}_I(G) \cong \mathbf{Vect}$, where $\mathbf{Vect}$ is the category of Hermitian inner product spaces discussed above, and hence its K–theory spectrum is precisely $\mathbf{ku}$. The inclusion i of this subcategory into $\widetilde{\mathcal{R}}(G)$ is an equivalence of permutative categories, with inverse the continuous permutative functor $q : \widetilde{\mathcal{R}}(G) \to \widetilde{\mathcal{R}}_I(G)$ defined by $q(\rho) = I_{\dim \rho}$ and $q(\rho, \psi, A) = (I_{\dim \rho}, I_{\dim \psi}, A)$ (note that the domain and codomain of any morphism in $\widetilde{\mathcal{R}}(G)$ have the same dimension). The composition iq is the identity functor on $\mathbf{Vect}$, while the identity matrices $I_n \in U(n)$ yield a natural isomorphism $\nu : qi \to \mathrm{Id}_{\widetilde{\mathcal{R}}(\Gamma)}$ (that is, $\nu_\rho = I_{\dim \rho}$).

It is a basic fact that continuous permutative functors induce maps of K–theory spectra. Hence the maps i and q induce maps $\mathbb{K}(i)$ and $\mathbb{K}(q)$ between $\mathbb{K}(\widetilde{\mathcal{R}}(G))$ and $\mathbf{ku} = \mathbb{K}(\mathbf{Vect})$, and we want to verify that maps are weak equivalences. We have $\mathbb{K}(i) \circ \mathbb{K}(q) = \mathbb{K}(iq) = \mathrm{Id}_{\mathbf{Vect}}$. The above natural transformation shows that the map on *classifying spaces* induced by the functor qi is homotopic to the identity. The zeroth space of the K–theory spectrum of a permutative category $\mathcal{P}$ is simply $\Omega B|\mathcal{P}|$ (this follows from [12, Construction 10, Step 2]), so we conclude that on zeroth spaces, the map $\mathbb{K}(qi)$ is homotopic to the identity. Since K–theory spectra are connective, it follows that i and q induce inverse isomorphisms on homotopy groups, and hence are weak equivalences of spectra. By Proposition 3.3, the map

$$\mathbf{ku}^{hG} \longrightarrow \mathbb{K}(\widetilde{\mathcal{R}}(\Gamma))^{hG}$$

induced by i is a weak equivalence as well.

To complete the proof, we just need to check that $\mathbf{ku}^{hG} = F^G(EG, \mathbf{ku})$ is equivalent to $F(BG, \mathbf{ku})$. Since G acts *trivially* on the subcategory $\widetilde{\mathcal{R}}_I(G) \cong \mathbf{Vect}$, its action on $\mathbf{ku} \cong \mathbb{K}(\mathbf{Vect})$ is trivial and we have $F^G(EG, \mathbf{ku}) \cong F(EG/G, \mathbf{ku}) \cong F(BG, \mathbf{ku})$. □

We now have natural maps of spectra

$$K^{\mathrm{def}}(G) \xrightarrow{\cong} \mathbb{K}(\widetilde{\mathcal{R}}(G))^G \hookrightarrow \mathbb{K}(\widetilde{\mathcal{R}}(G))^{hG} \xleftarrow{\simeq} \mathbb{K}(\widetilde{\mathcal{R}}_I(G))^{hG} \xleftarrow{\cong} F(BG_+, \mathbf{ku}),$$

each natural in the group G. After passing to the derived category of spectra, we obtain the desired natural transformation

$$K^{\mathrm{def}}(G) \longrightarrow F(BG_+, \mathbf{ku}).$$

It follows from work of Lawson [10] that the spectrum $\mathbf{ku}$ can be rigidified to a commutative $\mathbf{S}$–algebra $\mathbf{ku}^r$ in the sense of Elmendorff–Kriz–Mandell–May [7], and moreover $K^{\mathrm{def}}(G)$ and $F(BG_+, \mathbf{ku})$ can be rigidified to modules over this $\mathbf{S}$–algebra. The above map then corresponds to a map in the derived category of $\mathbf{ku}^r$–algebras.

Lawson has given a rather different construction of the spectrum $K^{\mathrm{def}}(G)$, in which the ring structure arises from an external pairing of

universes. It is known that additively (i.e. as spectra) Lawson's model agrees with the model presented here (based on permutative categories), but the comparison goes through a third model for which it seems more difficult to construct a ring structure. The following (incomplete) argument (Proposition 3.5) outlines how one might show that the ring spectra produced by Lawson are equivalent (as rings) to the model used here.

The need for such a comparison is simply that we will use Lawson's product formula below, which he proved using his model for $K^{\mathrm{def}}(G)$. A different approach, which seems simpler, is to instead give a direct proof of the product formula for the (bi)permutative model of $K^{\mathrm{def}}(G)$. The product formula states that $K^{\mathrm{def}}(G \times H) \simeq K^{\mathrm{def}}(G) \wedge_{\mathbf{ku}} K^{\mathrm{def}}(H)$ (as rings), and Lawson's proof proceeds by filtering both sides by rank and showing that natural map induces weak equivalences on the filtration quotients. The additive comparisons between the two models respect these filtrations, and hence can be used to deduce the product formula for the (bi)permutative model from Lawson's version. It should be noted that the map $K^{\mathrm{def}}(G) \wedge_{\mathbf{ku}} K^{\mathrm{def}}(H) \to K^{\mathrm{def}}(G \times H)$ is induced by functoriality, and is automatically a ring map for either model.

Proposition 3.5. *The natural map $K^{\mathrm{def}}(G) \to F(BG_+, \mathbf{ku})$ corresponds to a map $K^{\mathrm{def}}(G)^r \to F(BG_+, \mathbf{ku}^r)$ in the derived category of $\mathbf{ku}^r$–algebras.*

Proof. (Sketch) The spectra $\mathbb{K}(\mathrm{R}(G)) = K^{\mathrm{def}}(G)$, $\mathbb{K}(\widetilde{\mathcal{R}}(G))$, and $\mathbf{ku}$ can be rigidified by taking the simplicial permutative categories of singular simplices in the underlying categories. These simplicial categories are in fact simplicial *bipermutative* categories under tensor product of representations and unitary matrices. The results of Elmendorff and Mandell [8] now show that the associated simplicial K–theory spectrum is equivalent to a simplicial object in the category of commutative ring symmetric spectra, and then apply Schwede's functor from symmetric spectra to $\mathbf{S}$–algebras.

It remains to show that this rigidification process produces $\mathbf{S}$–algebras that are equivalent, in the derived category of $\mathbf{S}$–algebras, to the model used by Lawson. In [16], a comparison between the spectrum $K^{\mathrm{def}}(\mathrm{R}(G))$ and Lawson's explicit Γ–space model for deformation K–theory was given. The intermediate spectrum is the spectrum associated to the monoid

$$\coprod_n \mathrm{Hom}(\Gamma, U(n)) \times_{U(n)} (EU(n) \times V(n)), \tag{4}$$

where $EU(n)$ denotes the categorical model and $V(n)$ is the Stiefel manifold on n–frames in $\mathbb{C}^\infty$, and projecting onto the $EU(n)$ and the $V(n)$ factors gives equivalences from this model to the models for $K^{\mathrm{def}}(G)$ used here and in Lawson's work, respectively. These equivalences passes to equivalences between the associated singular objects. The technical part of this proof will be to check that the monoid (4) gives rise to an E_∞ ring spectrum, and that the comparison maps respects the multiplications. Roughly speaking, the multiplication on (4) is induced by tensor product of representations in the Hom factor, Kronecker product of unitary matrices in the $U(n)$ factor, and the tensor product map

$$V(n) \times V(m) \to V(nm)$$

given by tensoring an n–frame with an m–frame to obtain an nm–frame in $\mathbb{C}^\infty \otimes \mathbb{C}^\infty$ (with Hermitian inner product induced by $\langle v \otimes w, v' \otimes w' \rangle = \langle v, v' \rangle \langle w, w' \rangle$) and then applying an isomorphism $\mathbb{C}^\infty \otimes \mathbb{C}^\infty \cong \mathbb{C}^\infty$.

The final step is to compare Lawson's rigidification of his model for $K^{\mathrm{def}}(G)$ with the analogous rigidification of the singular object associated to Lawson's model. This can be done using the results on simplicial spectra in [7]. □

4. The homotopy limit problem and maps between classifying spaces

In this section we compare the natural map

$$K^{\mathrm{def}}(G) \longrightarrow F(BG_+, \mathbf{ku})$$

constructed in Section 3 with the maps

$$B_n \colon \operatorname{Hom}(G, U(n)) \longrightarrow \operatorname{Map}(BG, BU(n))$$

sending representations to the induced maps between classifying spaces.

To begin, we define two functors associated to a representation $\rho \colon G \to U(n)$. Let EG denote the translation category of G; that is, EG has an object for each $g \in G$ and a unique morphism $g \to h$ for each $g, h \in G$ (we write this morphism simply as $g \to h$). We will also use the notation EG to denote the geometric realization of this category; recall that EG is a model for the universal principal G–bundle, via the action of G on EG defined by *right* multiplication, and the quotient EG/G is (the realization of) the category BG with a single object $*$ and $\operatorname{Mor}_{BG}(*, *) = G$, with composition given by multiplication (that is, $g \circ h = gh$). We define

$$C_\rho \colon EG \to \mathrm{R}(G) \tag{5}$$

to be the constant functor at the object $\rho \in \mathrm{R}(G)$; that is, $C_\rho(g) = \rho$ and $C_\rho(g \to h) = I_n$ for all $g, h \in G$. Next, we define the functor

$$B_\rho \colon EG \to \mathrm{R}(G) \tag{6}$$

by $B_\rho(g) = I_n$, where I_n denotes the trivial representation $G \to U(n)$, and $B_\rho(g \to h) = \rho(hg^{-1})$.

We can also define parametrized versions of the functors C_ρ and B_ρ. Given a space X and a mapping $\rho : X \to \operatorname{Ob}(\mathrm{R}(G)) = \coprod_n \operatorname{Hom}(G, U(n))$, we define the maps

$$C_\rho, B_\rho \colon X \times EG \longrightarrow |\mathrm{R}(G)|$$

to be the adjoints of the maps $X \to F(EG_+, \mathrm{R}(G))$ given by $x \mapsto C_{\rho(x)}$ and $x \mapsto B_{\rho(x)}$ respectively. To see that C_ρ and B_ρ are continuous, it suffices to consider the universal example, in which $X = \operatorname{Ob}\mathrm{R}(G)$ and ρ is the identity map. Then we have continuous functors

$$\operatorname{Ob}\mathrm{R}(G) \times EG \to \mathrm{R}(G)$$

(considering $\operatorname{Ob}\mathrm{R}(G)$ as a category with only identity morphisms) whose realizations yield C_ρ and B_ρ, which must then be continuous. (Note that continuity of B_ρ depends on continuity of the evaluation map $\operatorname{Hom}(G, U(n)) \times G \to U(n)$.)

Lemma 4.1. *For any map $\rho\colon X \to \mathrm{Hom}(G, U(n))$, the maps C_ρ and B_ρ are G–equivariantly homotopic as maps*

$$X \times EG \longrightarrow \mathrm{R}(G) \hookrightarrow \widetilde{\mathcal{R}}(G).$$

Proof. For each $\psi \in \mathrm{Hom}(G, U(n))$, there is a natural transformation $\eta\colon C_\psi \to B_\psi$ defined by $\eta_g = \psi(g)$. (Note that $\psi(g) \in U(n)$ is a morphism in $\widetilde{\mathcal{R}}(G)$ from $C_\psi(g) = \psi$ to $B_\psi(g) = I_n$, but this morphism is *not* equivariant, and hence does not live in the smaller category $\mathrm{R}(G)$.) The fact that η is a natural between functors out of EG is summarized by the following commutative diagram in $\widetilde{\mathcal{R}}(G)$, associated to a morphism $g \to h$ in EG:

$$\begin{array}{ccc} C_\psi(g) = \psi & \xrightarrow{\eta_g = \psi(g)} & B_\psi(g) = I_n \\ {\scriptstyle C_\psi(g\to h) = I_n}\downarrow & & \downarrow{\scriptstyle B_\psi(g\to h) = \psi(hg^{-1})} \\ C_\psi(h) = \psi & \xrightarrow[\eta_h = \psi(h)]{} & B_\psi(h) = I_n. \end{array}$$

Now, given a map $\rho\colon X \to \mathrm{Hom}(G, U(n))$, we obtain a homotopy

$$X \times EG \times I \longrightarrow \widetilde{\mathcal{R}}(G) \tag{7}$$

between C_ρ and B_ρ as the adjoint of the map

$$X \longrightarrow F((EG \times I)_+, \widetilde{\mathcal{R}}(G))$$

defined by $x \mapsto \eta_{\rho(x)}$. Continuity can be checked in the same way as for C_ρ and B_ρ, noting that a natural transformation between functors $f, g : \mathcal{C} \to \mathcal{D}$ can be viewed as a functor $\mathcal{C} \times I \to \mathcal{D}$, where I is the category with objects $\{0, 1\}$ and a unique morphism $0 \to 1$. It is an exercise in the definitions to check that the functor $EG \times I \to \widetilde{\mathcal{R}}(G)$ corresponding to η is G–equivariant, and it follows that the homotopy (7) is G–equivariant. □

We need to review the McDuff–Segal approach to group completion [13]. Given a topological monoid M with multiplication $(m, n) \mapsto m * n$ and an element $m_0 \in M$, let $M_\infty(m_0)$ denote the infinite mapping telescope of the sequence

$$M \xrightarrow{*m_0} M \xrightarrow{*m_0} M \xrightarrow{*m_0} \cdots,$$

where $*m_0$ denotes the map $m \mapsto m * m_0$. McDuff and Segal construct a weakly contractible space $(M_\infty(m_0))_M$ and a map $q_{m_0}(M_\infty(m_0))_M \to BM$ whose fiber over the basepoint $* \in BM$ is precisely $M_\infty(m_0)$. If m_0 satisfies the condition that for all $m \in M$, there exists $n \in M$ and $k \in \mathbb{N}$ such that $m * n$ lies in the connected component of

$$m_0^k := \overbrace{m_0 * \cdots * m_0}^{k},$$

then we say that M is *stably group-like* with respect to m_0. If, furthermore, there is a homotopy $H\colon M \times M \times I \to M$ such that $H_0(m, n) = m * n$, $H_1(m, n) = n * m$, and $H_t(m_0^k, m_0^l) = m_0^{k+l}$ for all $t \in [0, 1]$, then natural map

$$M_\infty(m_0) \hookrightarrow \mathrm{hofib}(q_{m_0})$$

is a weak equivalence by the proof of Ramras [15, Theorem 3.6]. The existence of the homotopy H for the monoids considered here depends on some linear algebra, explained in [15, §4].

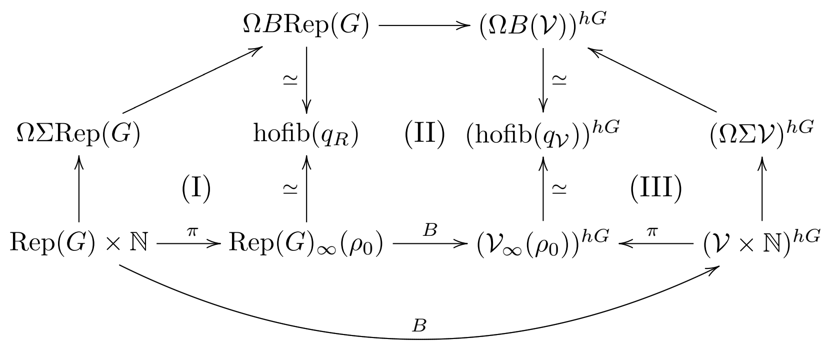
The realization of the category $\mathrm{R}(G)$ is the monoid of homotopy orbit spaces

$$\mathrm{Rep}(G)_{hU} := \coprod_n \mathrm{Hom}(G, U(n))_{hU(n)}$$

(for a proof, see Ramras [15, Proposition 2.4] or Lawson [9, §6.2]).

For the remainder of this section, we assume that the monoid $\mathrm{Rep}(G)$ is stably group-like with respect to a representation $\rho_0 \in \mathrm{Hom}(G, U(n))$ $(n > 0)$. Let $\mathcal{V}$ denote the realization of the category $\widetilde{\mathcal{R}}(G)$, and note that by Lemma 3.4, $\mathcal{V}$ is homotopy equivalent to the geometric realization of **Vect**, which is the space $\coprod_n BU(n)$. It follows that $\pi_0 \mathcal{V} \cong \mathbb{N}$, so $\mathcal{V}$ is stably group-like with respect to any of its non-zero elements, and in particular the object $\rho_0 \in \widetilde{\mathcal{R}}(G)$. We denote the map $q_{\rho_0} \colon \mathrm{Rep}(G)_\infty(\rho_0) \to B\mathrm{Rep}(G)$ by q_R and we denote the map $q_{\rho_0} \colon (\mathcal{V}_\infty(\rho_0))_{\mathcal{V}} \to B\mathcal{V}$ by $q_{\mathcal{V}}$. In order to compare the natural map $K^{\mathrm{def}}(G) \to F(BG_+, \mathbf{ku})$ to the maps $B \colon \mathrm{Hom}(G, U(n)) \to \mathrm{Map}(BG, BU(n))$, we consider the following diagram, whose various maps are explained below.

(8)



Our goal will be to show that the rectangle (II) commutes, at least on homotopy groups, since this will relate the natural map $K^{\mathrm{def}}(G) \to F(BG_+, \mathbf{ku})$, which is quite closely related to the top map, to the bottom map B, which (as we will explain) is just built from the maps $B \colon \mathrm{Hom}(G, U(n)) \to F(BG_+, BU(n))$. We begin by explaining the various maps in this diagram.

The top map is the restriction to $\mathrm{Rep}(G)$ of the map on zeroth spaces induced by the map of spectra $K^{\mathrm{def}}(G) \to F(BG_+, \mathbf{ku})$ (or more precisely, it is the restriction of the map on zeroth spaces induced by $K^{\mathrm{def}}(G) = \mathbb{K}(\widetilde{\mathcal{R}}(G))^G \hookrightarrow \mathbb{K}(\widetilde{\mathcal{R}}(G))^{hG}$). Note that $\mathrm{Rep}(G)$ is the monoid of objects in $\mathrm{R}(G)$, and this gives inclusions $\mathrm{Rep}(G) \hookrightarrow |\mathrm{R}(G)|$ and $\Omega B\mathrm{Rep}(G) \hookrightarrow \Omega B|\mathrm{R}(G)|$. The vertical weak equivalences are those arising from the McDuff–Segal group completion picture, discussed above. The maps labeled π send the n^{th} copy of the relevant monoid to the n^{th} stage of the mapping telescope. The polygons (I) and (III) require some further explanation. The maps labelled B are induced by the maps

$$\mathrm{Hom}(G, U(n)) \longrightarrow \mathrm{Map}(BG, \mathcal{V})$$

whose adjoints

$$\mathrm{Hom}(G, U(n)) \times BG \to \mathcal{V}$$

are induced by the maps B_ρ from Lemma 4.1. Note that for each $\rho \in \mathrm{Hom}(G, U(n))$, the functor $B_\rho \colon EG \to \widetilde{\mathcal{R}}(G)$ factors through the projection functor $EG \to BG$, inducing the functorial map $B\rho : BG \to BU(n)$, where $BU(n)$ is viewed as the automorphisms of the object $I_n \in \mathrm{Ob}\,\widetilde{\mathcal{R}}(G)$. It follows that the bottom portion of the diagram is commutative, that is, $\pi \circ B = B \circ \pi \colon \mathrm{Rep}(G) \times \mathbb{N} \to (\mathcal{V}_\infty(\rho_0))^{hG}$.

Consider a topological monoid M and an element $m \in M$. Then there is a natural map $\Sigma M \to BM$, resulting from the fact that the space of 1-simplices in BM is precisely M (note here that Σ denotes the *reduced* suspension, with the identity element $e \in M$ as basepoint). Furthermore, we have a map $M \to \Omega\Sigma M$ given by sending $x \in M$ to the loop

$$\alpha_x(t) = [(x,t)] \in \Sigma M = (M \times I)/\sim .$$

We now define a map $M \times \mathbb{N} \to \Omega\Sigma M$ by sending (x, n) to the composite loop $\alpha_{m^n}^{-1} \cdot \alpha_x$. (We use the convention that $\gamma \cdot \gamma'$ represents the loop which traverses γ for $t \in [0, 1/2]$ and γ' for $t \in [1/2, 1]$.) We can now consider the diagram

$$\begin{array}{ccc} & & \Omega BM \\ & \nearrow & \downarrow \\ \Omega\Sigma M & & \mathrm{hofib}(q_m) \\ \uparrow & & \uparrow \\ M \times \mathbb{N} & \xrightarrow{\pi} & M_\infty(m) \end{array} \tag{9}$$

Polygon (I) in Diagram 8 is created in exactly this manner, with $M = \mathrm{Rep}(G)$. Polygon (III) is created by applying this process to $\mathcal{V}$ and then forming the induced diagram on homotopy fixed sets; this makes sense since whenever G acts on M by monoid maps, the maps in Diagram (9) are G–equivariant.

Lemma 4.2. *Diagram 9 is* homotopy commutative. *Furthermore, if G acts on M by monoid maps, then this homotopy is a homotopy through G–equivariant maps* $M \times \mathbb{N} \to \mathrm{hofib}(q_m)$. *Consequently, the polygons (I) and (III) in Diagram 8 are homotopy commutative.*

This result follows by (a parametrized version of) the argument at the end of the proof of Ramras [15, Theorem 3.6]. (The argument given in that paper simply showed that the Diagram 9 induces a commutative diagram after passing to path components; extending that argument to prove the lemma is routine.)

We have now shown that polygons (I) and (III), as well as the lower portion of Diagram 8, are (homotopy) commutative. We claim that the outermost circuit of the diagram is homotopy commutative as well; that is, the two maps $\mathrm{Rep}(G) \times \mathbb{N} \to (\Omega B(\mathcal{V}))^{hG}$ are homotopic. This follows quite easily from Lemma 4.1.

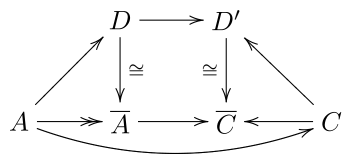
Proposition 4.3. *For any $k \geqslant 0$, the diagram*

$$\begin{array}{ccc} \pi_k \Omega B\mathrm{Rep}(G) & \longrightarrow & \pi_k(\Omega B(\mathcal{V}))^{hG} \\ \simeq \downarrow & & \downarrow \simeq \\ \pi_k \mathrm{hofib}(q_R) & & \pi_k(\mathrm{hofib}(q_{\mathcal{V}}))^{hG} \\ \simeq \uparrow & & \uparrow \simeq \\ \pi_k \mathrm{Rep}(G)_\infty(\rho_0) & \xrightarrow{B} & \pi_k(\mathcal{V}_\infty(\rho_0))^{hG} \end{array} \tag{10}$$

is commutative.

Proof. The basic idea is that in any diagram of groups the form

(11)



in which the map $A \to \overline{A}$ is surjective and both triangles, the outer circuit (from A to D') and the lower circuit (from A and $\overline{C}$) are commutative, the middle square must be commutative as well. (This follows from a simple diagram chase.)

In order to apply this to Diagram (8) and deduce the commutativity of (10), we set A and C to be the direct sums

$$\bigoplus_{n\in\mathbb{N}} \pi_k(\mathrm{Rep}(G) \times \{n\}, \rho_0^n) \text{ and } \bigoplus_{n\in\mathbb{N}} \pi_k(\mathcal{V} \times \{n\}, \rho_0^n)$$

respectively. The vertical maps on either side of Diagram (8) send (ρ_0^n, n) to the loop $\alpha_{\rho_0^n}^{-1} \cdot \alpha_{\rho_0^n}$, which is canonically homotopic to the constant loop at the basepoint. So we have canonical maps from these direct sums to the homotopy groups of $\Omega B\mathrm{Rep}(G)$ and $\Omega B\mathcal{V}$ (based at the constant loop at the basepoint of the classifying space). Similarly, the maps labelled π send (ρ_0^n, n) to the point ρ_0^n in the n^{th} stage of the relevant mapping telescope, and there is a canonical path from this point to the basepoint $(0,0)$ of the telescope (here $(0,0)$ represents the formal unit $0 \in \mathrm{Hom}(G, U(n))$ at the initial stage of the telescope). The map

$$\bigoplus_{n\in\mathbb{N}} \pi_k(\mathrm{Rep}(G) \times \{n\}, \rho_0^n) \longrightarrow \pi_k \mathrm{Rep}(G)_\infty(\rho_0)$$

is surjective; more precisely, this map induces an isomorphism

$$\operatorname*{colim}_{n\to\infty} \pi_k(\mathrm{Rep}(G) \times \{n\}, \rho_0^n) \xrightarrow{\cong} \pi_k \mathrm{Rep}(G)_\infty(\rho_0).$$

The result now follows from general observation in the first paragraph. (Note that the diagrams still commute despite our adjustments to the basepoints. This follows from the fact that all the fundamental groups of the spaces appearing in Diagram (8) are abelian, since these spaces are infinite loop spaces.) □

Proposition 4.4. *Assume that* $\mathrm{Rep}(G)$ *is stably group-like with respect to the trivial one-dimensional representation. The natural map* $K^{\mathrm{def}}(G) \to F(BG_+, \mathbf{ku})$ *is an isomorphism (respectively, a surjection or an injection) on* π_k *if and only if the map*

$$B\colon \mathrm{Hom}(G, U) \to \mathrm{Map}_*(BG, BU)$$

is an isomorphism (respectively, a surjection or an injection) on π_k *(*$k \geqslant 0$*).*

Proof. We need to consider the a version of Diagram (8) in which the monoid $\mathrm{Rep}(G)$ is replaced by larger monoid $\mathrm{Rep}(G)_{hU}$, since then the top map in the diagram,

$$\Omega B\mathrm{Rep}(G)_{hU} \to (\Omega B\mathcal{V})^{hG},$$

becomes the map on zeroth spaces induced by

$$K^{\mathrm{def}}(G) \to (K^{\mathrm{def}}(\widetilde{\mathcal{R}}(G)))^{hG} \simeq F(BG_+, \mathbf{ku}).$$

Rather than trying to define a natural mapping out of $\mathrm{Rep}(G)_{hU}$ generalizing the map $B : \mathrm{Rep}(G) \to F(BG, \mathbf{ku})$, we simply observe that $\mathrm{Rep}(G)_{hU}$ fibers over $\coprod_n BU(n)$ with $\mathrm{Rep}(G)$ as fiber, and moreover this fibration *splits*, via the maps $BU(n) \to \mathrm{Hom}(G, U(n))_{hU(n)}$, $x \mapsto [I_n, \tilde{x}]$, where $\tilde{x}$ is any lift of x to $EU(n)$ (continuity of this splitting follows from the fact that $EU(n) \to BU(n)$ is locally trivial). Similarly, $(\mathrm{Rep}(G)_{hU})_\infty(\rho_0)$ fibers over $\mathrm{colim}_n BU(n)$ with fiber $\mathrm{Rep}(G)_\infty(\rho_0)$. Hence the homotopy groups $\pi_k\left(\mathrm{Rep}(G)_{hU}, [\rho_0^n, *]\right)$ split as direct sums $\pi_k(\mathrm{Rep}(G), \rho_0^n) \oplus \pi_k BU(\dim(\rho_0^n))$. One has an analogous fibering relating

$$(\mathcal{V}_\infty(\rho_0))^{hG} \simeq \mathrm{Map}(BG, \Omega B(\coprod_n BU(n))$$

to

$$\mathrm{Map}_*\left(BG, \Omega B\left(\coprod_n BU(n)\right)\right)$$

and

$$\Omega B\left(\coprod_n BU(n)\right) \simeq \mathbb{Z} \times \mathrm{colim}\left(\coprod_n BU(n) \xrightarrow{\oplus \epsilon^{\dim(\rho_0)}} \coprod_n BU(n) \xrightarrow{\oplus \epsilon^{\dim(\rho_0)}} \cdots\right)$$

and another relating

$$(\mathcal{V} \times \mathbb{N})^{hG} = \coprod_n \mathrm{Map}(BG, BU(n))$$

to $\coprod_n \mathrm{Map}_*(BG, BU(n))$ and $\coprod_n BU(n)$. Consequently, we may extend Diagram (8) to a diagram of *homotopy groups* based on the monoid $\mathrm{Rep}(G)_{hU}$ simply by letting the maps corresponding to B be the direct sums of the maps in Diagram (8) and the identity maps on $\pi_* BU(n)$ or $\pi_* \mathbb{Z} \times \mathrm{colim}_n BU(n)$.

We can now extend Proposition 4.3 to this larger diagram, and the desired conclusion follows immediately. □

The method in Baird–Ramras [3, Section 4] **can be used to extend the ideas in the previous proof to arbitrary finitely generated groups. The basic idea is that** $\mathrm{Rep}(G)$ **is the filtered colimit, over** $n \in \mathbb{N}$**, of the submonoids generated by representations of dimension at most** n**, and each of these is stably group-like with respect to** I_n **so long as** G **is finitely generated. In that paper (see Proposition 4.19), the topological Atiyah–Segal map was shown to bear a similar relationship to the map** B**. Combining these ideas leads to the conclusion that the topological Atiyah–Segal map agrees with the map on homotopy induced by the homotopy limit problem.**

5. Products of groups

In this section, we examine compare the homotopy limit problem for a direct product of groups to the homotopy limit problems for the factors.

Proposition 5.1. *Let G and H be finitely generated groups such that BG and BH are homotopy equivalent to finite CW complexes. Moreover, assume that $\pi_* K^{\mathrm{def}}(G)$ and $\pi_* K^{\mathrm{def}}(H)$ are finitely generated (abelian) groups for each $*$, and that the Bott* *maps β_G and β_H induce split injections on homotopy in each dimension. Under these condition, if the natural map $K^{\mathrm{def}}(G) \to F(BG_+, \mathbf{ku})$ induces an isomorphism on π_* for $* \geqslant k$, and the natural map $K^{\mathrm{def}}(H) \to F(BH_+, \mathbf{ku})$ induces an isomorphism on π_* for $* \geqslant l$, then the natural map $K^{\mathrm{def}}(G \times H) \to F(B(G \times H)_+, \mathbf{ku})$ induces an isomorphism on π_* for $* \geqslant k + l + 1$.*

We note that the hypotheses of this theorem apply to all surface groups (Ramras [17, Theorem 6.1]), and to free groups (Lawson [11]). Also, this result fits with the expectation that for many groups, the from deformation K–theory to topological K–theory will be an equivalence above the rational cohomological dimension of the group, minus 2. Let us say, for the moment, that G is *good* if $K^{\mathrm{def}}(G \times H) \to F(B(G \times H)_+, \mathbf{ku})$ induces an isomorphism on π_* for $* > \mathbb{Q}\mathrm{cd}(G) - 2$. Say G and H are good. Assuming its other conditions are met, Proposition 5.1 tells us that $K^{\mathrm{def}}(G \times H) \to F(B(G \times H)_+, \mathbf{ku})$ induces an isomorphism on π_* for $* > (\mathbb{Q}\mathrm{cd}(G) - 2) + \mathbb{Q}\mathrm{cd}(H) - 2) + 2 = \mathbb{Q}\mathrm{cd}(G \times H) - 2$, so then $G \times H$ is good as well.

Proof. **This proof is incomplete: at the end, I need to assume that $F(BG_+, \mathbf{ku})$ and $F(BH_+, \mathbf{ku})$ have a relatively simple structure as ku–modules. This holds when G and H are products of aspherical *orientable* surface groups (or free groups) because then BG and BH are (stably) wedges of spheres. For non-orientable surfaces some additional argument is needed.**

The projections $G \leftarrow G \times H \to H$ induce maps on K^{def} in the opposite directions, and composing with the multiplication map μ for the $\mathbf{ku}$–algebra $K^{\mathrm{def}}(G \times H)$, we obtain a mapping

$$K^{\mathrm{def}}(G) \wedge_{\mathbf{ku}} K^{\mathrm{def}}(H) \longrightarrow K^{\mathrm{def}}(G \times H) \wedge_{\mathbf{ku}} K^{\mathrm{def}}(G \times H) \xrightarrow{\mu} K^{\mathrm{def}}(G \times H), \tag{12}$$

which is an equivalence of $\mathbf{ku}$–algebras (this is Lawson's Product Formula [10]).

To simplify notation, let $\Gamma = G \times H$, and note that $BG_+ \wedge BH_+ = (BG \times BH)_+ \simeq B\Gamma_+$. A construction for function spaces analogous to the product map (12) now yields a commutative diagram

$$\begin{array}{ccc}
K^{\mathrm{def}}(G) \wedge_{\mathbf{ku}} K^{\mathrm{def}}(H) & \xrightarrow{\alpha_G \wedge \alpha_H} & F(BG_+, \mathbf{ku}) \wedge_{\mathbf{ku}} F(BH_+, \mathbf{ku}) \\
\downarrow & & \downarrow \\
K^{\mathrm{def}}(\Gamma) \wedge_{\mathbf{ku}} K^{\mathrm{def}}(\Gamma) & \xrightarrow{\alpha_\Gamma \wedge \alpha_\Gamma} & F(B\Gamma_+, \mathbf{ku}) \wedge_{\mathbf{ku}} F(B\Gamma_+, \mathbf{ku}) \\
\downarrow \mu & & \downarrow \mu \\
K^{\mathrm{def}}(\Gamma) & \xrightarrow{\alpha_\Gamma} & F(B\Gamma_+, \mathbf{ku}).
\end{array} \tag{13}$$

Commutativity of this diagram follows from the fact that α is a natural transformation of functors from discrete groups to $\mathbf{ku}$–algebras. Note here that for any based space X, the $\mathbf{ku}$–algebra structure on $F(X, \mathbf{ku})$ is induced by the map

$$F(X, \mathbf{ku}) \wedge F(X, \mathbf{ku}) \longrightarrow F(X \wedge X, \mathbf{ku} \wedge \mathbf{ku}) \xrightarrow{\mu_*} F(X \wedge X, \mathbf{ku}) \xrightarrow{\Delta^*} F(X, \mathbf{ku}),$$

where the first map is the external pairing of function spectra, the second is induced by multiplication in $\mathbf{ku}$, and the last is induced by the diagonal $\Delta \colon X \to X \wedge X$.

This map coequalizes the two maps

$$F(X, \mathbf{ku}) \wedge \mathbf{ku} \wedge F(X, \mathbf{ku}) \longrightarrow F(X, \mathbf{ku}) \wedge F(X, \mathbf{ku})$$

given by left and right multiplication in $\mathbf{ku}$, and hence induces the desired map out of $F(X, \mathbf{ku}) \wedge_{\mathbf{ku}} F(X, \mathbf{ku})$.

To complete the proof, we will show that the vertical composite on the right side of this diagram is an equivalence, and that the map $\alpha_G \wedge \alpha_H$ induces an isomorphism on homotopy groups in dimensions greater than $k + l + 2$.

First we consider the righ-hand side of (13). Let $X^* = F(X, \mathbb{S})$ denote the Spanier–Whitehead dual of X (where X is a finite based CW complex). We consider $F(X, \mathbb{S})$ to be the function spectrum from [7, Section I.1], which has the structure of an $\mathcal{L}$–spectrum since $\mathbb{S}$ is an $\mathcal{L}$–spectrum [7, Proposition I.4.4]. Note that we have a weak equivalence of $\mathcal{L}$–spectra $F(X, \mathbb{S}) \simeq F_{\mathbb{S}}(\Sigma^\infty X, \mathbb{S})$; this follows from [7, Proposition II.1.4, Theorem I.8.5], which, respectively, give an isomorphism

$$F_{\mathbb{S}}(\Sigma^\infty X, \mathbb{S}) \cong \mathbb{S} \wedge_{\mathcal{L}} F(X, \mathbb{S})$$

and a weak equivalence

$$\mathbb{S} \wedge_{\mathcal{L}} M \longrightarrow M$$

for arbitrary $\mathcal{L}$–spectra M.

If Y is another finite based CW complex, there is a natural equivalence $f\colon (X \wedge Y)^* \xrightarrow{\simeq} X^* \wedge Y^*$ (see, for example, Cohen [6, p. 73]). Hence if X and Y are *unbased* finite CW complexes, we have

$$(X \times Y_+)^* = (X_+ \wedge Y_+)^* \simeq (X_+)^* \wedge (Y_+)^*.$$

Moreover, for any finite based CW complex X, and any $\mathbb{S}$–module M, we have $F(X, M) \simeq X^* \wedge M$. This equivalence comes from the chain of equivalences

$$X^* \wedge M = F(X, \mathbb{S}) \wedge M \simeq F(X, \mathbb{S} \wedge M) \simeq F(X, M).$$

The equivalence $F(X, \mathbb{S}) \wedge R \simeq F(X, \mathbb{S} \wedge R)$ is guaranteed by [7, Theorem III.7.9], which states that the natural map

$$F(X, \mathbb{S}) \wedge M = F_{\mathbb{S}}(\Sigma^\infty X, \mathbb{S}) \wedge M \to F_{\mathbb{S}}(\Sigma^\infty X, \mathbb{S} \wedge M)$$

is an equivalence in the derived category of $\mathbb{S}$–modules, so long as $\Sigma^\infty X$ is weakly equivalent to a finite cell $\mathbb{S}$–module. In general, the suspension spectrum of a finite CW complex is a finite cell $\mathbb{S}$–module; this follows by induction from the natural weak equivalences

$$\mathbb{S}^n \simeq \mathbb{S} \wedge_{\mathcal{L}} \mathbb{S}^n \simeq \mathbb{S} \wedge_{\mathcal{L}} \mathbb{L}\mathbb{S}^n$$

(see [7, Proposition I.8.2 and Theorem I.4.6]).

Applying Spanier–Whitehead duality to the right-hand side of Diagram (13) yields the maps

$$\begin{aligned} &((BG_+)^* \wedge \mathbf{ku}) \wedge_{\mathbf{ku}} ((BH_+)^* \wedge \mathbf{ku}) \\ &\qquad \longrightarrow ((BG_+)^* \wedge (BH_+)^* \wedge \mathbf{ku}) \wedge_{\mathbf{ku}} ((BG_+)^* \wedge (BH_+)^* \wedge \mathbf{ku}) \\ &\quad \simeq ((B\Gamma_+)^* \wedge \mathbf{ku}) \wedge_{\mathbf{ku}} ((B\Gamma_+)^* \wedge \mathbf{ku}) \xrightarrow{\Delta^* \wedge \mu} ((BG \times BH)_+)^* \wedge \mathbf{ku} \end{aligned}$$

But this composite is the same as the map

$$((BG_+)^* \wedge \mathbf{ku}) \wedge_{\mathbf{ku}} ((BH_+)^* \wedge \mathbf{ku}) \xrightarrow{f \wedge \mu} ((BG \times BH)_+)^* \wedge \mathbf{ku},$$

which is an equivalence since both f and $\mu\colon \mathbf{ku} \wedge_{\mathbf{ku}} \mathbf{ku} \to \mathbf{ku}$ are equivalences.

The connectivity estimate for $\alpha_G \wedge \alpha_H$ now follows easily if the $K^{\mathrm{def}}(G)$, $K^{\mathrm{def}}(H)$, $F(BG_+, \mathbf{ku})$ and $F(BH_+, \mathbf{ku})$ are all wedges of cyclic $\mathbf{ku}$–modules. This is the case for the deformation K–theory spectra since we have assumed that the Bott map is a split injection and the homotopy groups are finitely generated (and for surface groups and free groups, it also follows from the explicit calculations in Ramras [17]). For the function spectra, this is the case if the domain space is (stably) a wedge of spheres. In general, I'm unsure how the Bott map behaves in negative degrees. I think it should suffice to sort out the case of $\mathbb{RP}^2$.

□

6. Vector bundles over products of surfaces

Note: This section needs to be rewritten in light of the results in Baird–Ramras [3], which show that the notion of flat family considered here is really the same as a family of bundles admitting a smoothly varying family of flat connections. Also, for the case of a single surface, the results below appear in [3, Section 3].

The results below are written without the orientability condition, although this may need to be added in light of the gap in Section 5.

Given a space X with universal cover $\widetilde{X}$ and a representation $\rho\colon \pi_1 X \to U(n)$, we can form the mixed bundle

$$E_\rho \to X$$

by setting $E_\rho = (\widetilde{X} \times \mathbb{C}^n)/\pi_1 X$, where $\pi_1 X$ acts diagonally via deck transformations of $\widetilde{X}$ and the representation ρ. We call such bundles *flat*, since when X is a manifold these bundles admit flat connections (whose holonomy representation is precisely ρ). We call an m–dimensional complex vector bundle $E \to X$ *stably flat* if for some $n \in \mathbb{N}$, the bundle $E \oplus \mathbb{C}^n$ is flat. Moreover, we call a vector bundle $E \to X \times S^k$ ($k > 0$) a *family of vector bundles* over X (parametrized by S^k) and we say that a family is flat if it has the form

$$(\widetilde{X} \times S^k \times \mathbb{C}^n)/\pi_1 X$$

where $\pi_1 X$ acts on $S^k \times \mathbb{C}^n$ via some family of representations

$$\rho\colon S^k \to \mathrm{Hom}(\pi_1 X, U(n)).$$

That is, $\gamma \in \pi_1 X$ acts on $Y \times \mathbb{C}^n$ by $\gamma \cdot (y, z) = (y, \rho_y(\gamma) z)$. Finally, we call a family of bundles over X a *stably flat* family if after adding some trivial family $\epsilon^n := X \times S^k \times \mathbb{C}^n$, it becomes a flat family.

In this section, we consider the following problem.

Problem 6.1. *Given a manifold M and family $E \to M \times S^k$ of complex vector bundles over M, when is E a (stably) flat family?*

There are two necessary conditions E must satisfy in order to be a flat family. First of all, the restriction of E to $M \times \{z\}$ must be flat for every $z \in S^k$; note that since S^k is path connected, these restrictions are isomorphic for different z and hence one is stably flat if and only if the others are. Second, we claim that the restriction of E to $\{m\} \times S^k$ must be trivial for every $m \in M$. Letting $f\colon M \to BG$

denote a classifying map for the universal cover of M, we know that if E is the flat family associated to $\rho\colon S^k \to \mathrm{Hom}(G, U(n))$, then E is classified by the composite

$$S^k \times M \xrightarrow{\mathrm{Id}\times f} S^k \times BG \xrightarrow{(B\rho)^\vee} BU(n).$$

Letting $\mathcal{E}_\rho = ((B\rho)^\vee)^*(EU(n))$, we see that the restriction of E to $S^k \times \{m\}$ is the pullback of $\mathcal{E}_\rho|_{S^k\times\{f(m)\}}$. However, $\mathcal{E}_\rho|_{S^k\times\{f(m)\}}$ is isomorphic to $\mathcal{E}_\rho|_{S^k\times\{x\}}$ for each $x \in BG$, and when x is the basepoint $* \in BG$, $\mathcal{E}_\rho|_{S^k\times\{*\}}$ is classified by the constant map to the basepoint $* \in BU(n)$. Hence if E is a flat family, its restriction to $S^k \times \{m\}$ must be trivial for each $m \in M$.

Similarly, if E is stably flat, then $E_{M\times\{z\}}$ must be stably flat for every $z \in S^k$, and $E|_{S^k\times\{m\}}$ must be stably trivial for each $m \in M$.

We note that in general, these conditions alone *do not* imply that E is (stably) flat. Let M be a $2l-k$–dimensional closed, orientable manifold, and consider a bundle $E \to S^k \wedge M$ with $c_l(E) = m[S^k]\otimes[M] \in H^2l(S^k \wedge M) \cong H^{2l-k}(M)$ for some $m \in \mathbb{Z}$ (note that such bundles exist because the Chern character is a rational isomorphism, and the cup product structure on a suspension is trivial). Letting E' denote the pullback of E to $S^k \times M$, we see that E' is trivial when restricted to either factor (since its classifying map factors through the smash product), and $c_l(E')$ is non-zero in rational cohomology. However, it is proven in [3] that the Chern classes of a flat family are rationally trivial above the dimension of the sphere parametrizing the family. Hence E' cannot be a flat family if $l > k$. More generally, we find that there is a third condition E must satisfy in order to be a stably flat family: its Chern classes must vanish rationally above the dimension of the parametrizing sphere. However, as we will see by analyzing the Heisenberg manifold, even this third condition is not enough to guarantee that E is stably flat.

Now consider the case where $M = S$ is a closed, aspherical surface (possibly non-orientable), and let $\widetilde{g}$ be the genus of the orientation double cover of S (which is just S if S is orientable). In this case, the answer to Problem 6.1 is that if $1 \leqslant k \leqslant (n-1)\widetilde{g}$, then *every* bundle over $S \times S^k$ which is flat when restricted to $M \times \{z\}$, and trivial over $\{m\} \times S^k$, is in fact flat. In Ramras [17, Theorem 3.4], it was shown using Morse theory for the Yang–Mills functional that the map

$$B\colon \mathrm{Hom}(\pi_1 S, U(n)) \to \mathrm{Map}_*(B\pi_1 M, BU(n)) \tag{14}$$

is (at least) $(1, (n-1)\widetilde{g})$–connected. This means that for all choices of basepoint, B induces isomorphism on π_* for $1 \leq * < (n-1)\widetilde{g}$, a surjection on $\pi_{\widetilde{g}}$, and an injection on π_0. Bundles $E \to S \times S^k \simeq B\pi_1 S \times S^k$ are classified by *unbased* maps $f\colon B\pi_1 S \times S^k \to BU(n)$. Under our assumption that E is trivial over $* \times S^k$, we know that there is a homotopy from $f|_{*\times S^k}$ to the constant map at the basepoint $* \in BU(n)$. Now, since $* \times S^k \hookrightarrow B\pi_1 S \times S^k$ is a CW inclusion (and in particular a cofibration) the Homotopy Extension Property allows us to extend this homotopy over all of $B\pi_1 S \times S^k$. At time 1, this homotopy gives us a map $S^k \to \mathrm{Map}_*(B\pi_1 S, BU(n))$ that still classifies E (note that this is still an unbased map out of S^k, though). So the bundles we are interested in are classified by unbased maps $S^k \to \mathrm{Map}_*(B\pi_1 S, BU(n))$ whose image lies in a component corresponding the flat bundles, i.e. a component in the image of the map (14).

The result from [17] quoted above says that

$$B\colon \mathrm{Hom}(\pi_1 S, U(n)) \to \mathrm{Map}_*(B\pi_1 M, BU(n))$$

is surjective on homotopy groups, or in other words on *based* maps

$$S^k \to \mathrm{Map}_*(B\pi_1 S, BU(n)).$$

In general, if a map $f\colon X \to Y$ induces isomorphisms on homotopy groups with respect to all basepoints, then it also induces bijections between these unbased mapping spaces (with an obvious caveat in case some path components of Y are disjoint from the image of f) . We record this observation in the next lemma.

Lemma 6.2. *Consider a map $f\colon X \to Y$, and let $Y_f \subset Y$ denote the union of the path components of Y that meet the image of f. If $f\colon X \to Y$ induces isomorphisms $f_*\colon \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(Y, f(x_0))$ for all $x_0 \in X$ and isomorphisms $f_*\colon \pi_k(X, x_0) \xrightarrow{\cong} \pi_n(Y, f(x_0))$ for all x_0 in X, then f induces a bijection*

$$[S^k, X] \xrightarrow{\cong} [S^k, Y_f]$$

between unbased homotopy classes of unbased maps.

Proof. Recall that for any based, path connected CW complex (Z, z_0) there is an action of $\pi_1(X, x_0)$ on the space $\langle Z, X\rangle$ of base-point preserving homotopy classes of basepoint preserving maps from Z to X, and the quotient of $\langle Z, X\rangle$ by this action is precisely the set of unbased homotopy classes of unbased maps from Z into the path component X_{x_0} of x_0. In our case, this says that $\pi_k(X, x_0)/\pi_1(X, x_0) \cong [S^k, X_{x_0}]$. The result now follows from the commutative diagrams

$$\begin{array}{ccc} \pi_k(X, x_0) & \xrightarrow{\cong} & \pi_k(Y, f(x_0)) \\ \downarrow{\scriptstyle /\pi_1(X,x_0)} & & \downarrow{\scriptstyle /\pi_1(Y,f(y_0))} \\ [S^k, X_{x_0}] & \longrightarrow & [S^k, Y_{f(y_0)}] \end{array} .$$

□

We can now see that every bundle $E \to S \times S^k$ that is flat in the S–direction and trivial in the S^k–direction is in fact flat (assuming $1 \leqslant k \leqslant (n-1)\widetilde{g}$): such bundles are classified by maps $S^k \to \mathrm{Map}_*(B\pi_1 S, BU(n))$ whose image is inside the image of B, and all such maps are in fact of the form $B\rho$ for some $\rho\colon S^k \to \mathrm{Hom}(\pi_1 S, U(n))$.

Next, consider the following closely related problem. Given a bundle $p\colon E \to S^k \times M$ such that $E|_{\{z\}\times M}$ is flat for each z, can we find a family of flat connections A_z that vary continuously with $z \in S^k$? (A flat connection on E can be viewed as a continuous mapping $A\colon p^*(T(S^k \times M)) \to T(E)$ splitting the natural map $\pi\colon TE \to p^*(T(S^k \times M))$, where now E denotes the principal $U(n)$–bundle associated to E, and by a continuous family of flat connections we simply mean a map $p^*(T_M(S^k \times M)) \to T(E)$, still splitting π, where $T_M(S^k \times M)$ denotes the sub-bundle of $T(S^k \times M)$ consisting of vectors tangent to M.) Note that if E is the flat family associated to $\rho\colon S^k \to \mathrm{Hom}(G, U(n))$, then the connections associated to the various representations $\rho(z)$ form such a continuous family.

When $M = S$ is a closed, aspherical surface, the only bundles $E \to S^k \times S$ with $E|_{\{z\}\times S}$ flat for each z that cannot be equipped with a continuous family of flat connections after stabilizing (in fact, the only ones which are not stably flat families) are those bundles formed from non-trivial bundles on S^k by pulling back along the projection $S^k \times S \to S^k$. This follows from the previous discussion, after examining the long exact sequence in homotopy associated to the split fibration

$$\mathrm{Map}_*(S, BU) \longrightarrow \mathrm{Map}(S, BU) \longrightarrow BU.$$

The bundles described above are those classified by maps $S^k \times S \to BU$ not in the image of $\pi_k \mathrm{Map}_*(S, BU)$, and all other bundles $E \to S^k \times S$ with $E|_{\{z\}\times S}$ flat for each z also have $E|_{S^k \times \{m\}}$ trivial for each $m \in S$, so that the previous discussion shows that they are stably flat.

We now use the results of the previous section to study Problem 6.1 for *products* of closed, aspherical surfaces. Note that if M is an aspherical manifold, then there is a homotopy equivalence $M \simeq B\pi_1 M$, and hence a weak equivalence of spectra

$$F(B\pi_1 M, \mathbf{ku}) \simeq F(M, \mathbf{ku}).$$

Once a universal cover of M is chosen, these maps become canonical: they are induced by the map $M \to B\pi_1 M$ classifying the universal cover as a principal $(\pi_1 M)$–bundle.

Proposition 6.3. *Let $M = S_1 \times \cdots \times S_k$ be a product of aspherical surfaces. Then the natural map*

$$K^{\mathrm{def}}(\pi_1 M) \to F(BM_+, \mathbf{ku})$$

induces an isomorphism on homotopy in dimensions greater than $\mathbb{Q}\mathrm{cd}(M) - 2$. Consequently, the natural map

$$B\colon \mathrm{Hom}(\pi_1 M, U) \longrightarrow \mathrm{Map}_*(M, BU)$$

induces an isomorphism on homotopy in dimensions greater than $\mathbb{Q}\mathrm{cd}(M) - 2$.

Proof. The monoid $\mathrm{Rep}(\pi_1 M)$ is stably group-like with respect to the trivial representation $1 \in \mathrm{Hom}(\pi_1 M, U(1))$ (Ramras [17, Lemma 6.4]), so we may set $\rho = 1$ when applying the results of the previous section.

When $k = 1$, this follows from Proposition 4.4 together with Ramras [17, Theorem 3.4]. One now extends to the case $k > 1$ by applying Proposition 5.1. The final statement follows by applying Proposition 4.4 once more. □

Reinterpreting the last statement in Proposition 6.3, we arrive at the following (partial) answer to Problem 6.1 for products of surfaces.

Corollary 6.4. *Let $M = S_1 \times \cdots S_k$ be a product of aspherical surfaces, and let $E \to M \times S^k$ be an m–dimensional complex vector bundle whose restriction to each $M \times \{z\}$ ($z \in S^k$) is flat, and whose restriction to $* \times S^k$ is trivial. If $k > \mathbb{Q}\mathrm{cd}(M)$, then E is stably flat, and the Chern classes $c_i(E)$ are torsion for $i > k$.*

Proof. We need to show that the isomorphism on homotopy groups based at the trivial representation actually leads to an isomorphism on homotopy groups at all basepoints. This follows from the fact that both spaces are (homotopy equivalent to) loop spaces, and hence loop multiplication provides homotopy equivalences between their components; moreover since B is a map of monoids it commutes with loop multiplication and hence induces isomorphisms on homotopy at all basepoints. (The loop spaces in question are $\Omega B\mathrm{Rep}(G)$ and $\Omega B\left(\coprod_n \mathrm{Map}(M, BU(n))\right)$; see Diagram 8.)

The consequences regarding Chern classes follow from Baird–Ramras [3, Theorem 3.3]. □

Question 6.5. *It seems like what I've said here is that in dimensions in which the cohomological obstructions are forced to vanish, we can always realize a point-wise flat family by a family of representations (modulo ruling out the above simple cases). One could hope, though, that the cohomological obstruction is really the only one, and that even in lower degrees all we have to assume about the bundle is that the appropriate Chern classes are rationally trivial. This would probably require a finer analysis of the map α for products of groups, so as to say something about it in low dimensions.*

7. Applications to crystallographic groups

Let Γ be virtually free abelian of finite rank; that is Γ has a finite index subgroup $T < \Gamma$ such that $T \cong \mathbb{Z}^k$. If T is normal and Γ/T acts faithfully on T, then Γ is crystallographic and we call T the *translation subgroup*. Using the transfer in complex K–theory, we can deduce rational information about the deformation K–theory of such groups.

Proposition 7.1. *If Γ be virtually free abelian of finite rank k, then the natural map*

$$K_*^{\mathrm{def}}(\Gamma) \otimes \mathbb{Q} \xrightarrow{\alpha_\Gamma \otimes \mathbb{Q}} K^{-*}(B\Gamma) \otimes \mathbb{Q}$$

is surjective for $* > k-2$.

Proof. Since $T \leqslant \Gamma$ has finite index, the map $B\Gamma = E\Gamma/\Gamma \xrightarrow{f} E\Gamma/T$ is a finite covering. Consider the commutative diagram

$$\begin{array}{ccccc}
K_*^{\mathrm{def}}(\Gamma) & \xrightarrow{\alpha_\Gamma} & K^{-*}(B\Gamma) & & \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle (Bi)^*} & \searrow{\scriptstyle f^*} & \\
K_*^{\mathrm{def}}(T) & \xrightarrow[\cong]{\alpha_T} & K^{-*}(BT) & \xleftarrow[\simeq]{\phi^*} & K^{-*}(E\Gamma/T)
\end{array}$$

where i is the inclusion $i\colon T \hookrightarrow \Gamma$ and ϕ is induced by $Ei\colon ET \to E\Gamma$. Recall there there is a transfer map $f_*\colon K^*(BT) \to K^*(B\Gamma)$ associated to the finite covering f, and this map satisfies $f_* \circ f^*(x) = [\Gamma : T]x = kx$ for all $x \in K^*(B\Gamma)$ (Becker–Gottlieb [4, Theorem 5.5]. This shows that $f_* \otimes 1_{\mathbb{Q}}$ is surjective (in general, if $h\colon A \to B$ is a map of abelian groups such that for all $b \in B$, there exists $n \in \mathbb{N}$ such that $nb \in \Im(h)$, then $h \otimes 1_{\mathbb{Q}} \to A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$ is surjective, because for any $(b, r/s)$, we know that $nb = h(a)$ for some $n \in \mathbb{N}$ and some $a \in A$ and now $h \otimes 1_{\mathbb{Q}}(a, r/(sn)) = (nb, r/(sn)) = (b, r/s)$). □

Proposition 7.2. *If Γ is a flat iterated torus bundle over a torus, then the natural map*

$$K_*^{\mathrm{def}}(\Gamma) \otimes \mathbb{Q} \xrightarrow{\alpha_\Gamma \otimes \mathbb{Q}} K^{-*}(B\Gamma) \otimes \mathbb{Q}$$

is an isomorphism for $* > k-2$.

Proof. We need to establish injectivity. Consider the commutative diagram

$$\begin{array}{ccc}
K_*^{\mathrm{def}}(\Gamma) & \xrightarrow{\alpha_\Gamma} & K^{-*}(B\Gamma) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle (Bi)^*} \\
K_*^{\mathrm{def}}(T) & \xrightarrow[\cong]{\alpha_T} & K^{-*}(BT)
\end{array}$$

It suffices to show that i^* is injective. This follows from an argument analogous to the proof of Ramras [18, Proposition 10.5]. Details will be added later.

□

8. Applications to spaces of flat connections

The results below are written without the orientability condition, although this may need to be added in light of the gap in Section 5.

In Baird–Ramras [3], it is shown that if $B\Gamma$ is homotopy equivalent to a smooth manifold M and $P \to M$ is a principal $U(n)$–bundle, then there is a weak equivalence

$$\mathcal{A}_{\text{flat}}(P) \simeq \text{hofib}\left(\text{Hom}(\Gamma, U(n)) \longrightarrow \text{Map}_*(B\Gamma, BU(n))\right),$$

where on the right the homotopy fiber is computed at the classifying map for P (which is well-defined up to homotopy if we fix a homotopy equivalence $M \to B\Gamma$). In this section, we combine our main results with this fact in order to derive results about $\pi_* \mathcal{A}_{\text{flat}}(P)$ when M is either a product of surfaces or a flat manifold.

In Baird–Ramras [3, Corollary 5.7], it was shown that if M is a d–dimensional aspherical manifold and $P \to M$ is a flat, principal $U(n)$–bundle, then the homotopy groups $\pi_*(\mathcal{A}_{\text{flat}}(P), A_0)$ are non-trivial in dimensions $m \leqslant \mathbb{Q}\text{cd}(M) - 3$, where $\mathbb{Q}\text{cd}(M)$ is the *rational cohomological dimension* (and A_0 is a arbitrary basepoint). If M is a surface, this result is vacuous, and it follows from Yang–Mills theory that the homotopy groups $\pi_*(\mathcal{A}_{\text{flat}}(P), A_0)$ for $* \leqslant 0 \leqslant f(n)$, where the upper bound $f(n)$ tends to infinity with n. Our results show that despite the non-trivial low-dimensional homotopy, this high-dimensional vanishing persists, in a sense, for products of surfaces.

Theorem 8.1. *If X is a product of aspherical surfaces and circles and $P \to X$ is a flat, principal $U(n)$–bundle, then for each $A_0 \in \mathcal{A}_{\text{flat}}(P)$ and each $* > \mathbb{Q}\text{cd}(X) - 3$, we have*

$$\pi_*(\mathcal{A}_{\text{flat}}(P), A_0) = 0.$$

The results of Section 7 lead to analogous results for flat iterated torus bundles over tori, at least rationally. **Precise statements will be added later.**

References

[1] M. F. Atiyah. Characters and cohomology of finite groups. *Inst. Hautes Études Sci. Publ. Math.*, (9):23–64, 1961.

[2] M. F. Atiyah and G. B. Segal. Equivariant K-theory and completion. *J. Differential Geometry*, 3:1–18, 1969.

[3] Thomas Baird and Daniel A. Ramras. Smoothing maps into algebraic set and the topological Atiyah–Segal transformation. arXiv:1206.3341, 2012.

[4] J. C. Becker and D. H. Gottlieb. The transfer map and fiber bundles. *Topology*, 14:1–12, 1975.

[5] Gunnar Carlsson. Equivariant stable homotopy and Segal's Burnside ring conjecture. *Ann. of Math. (2)*, 120(2):189–224, 1984.

[6] Joel M. Cohen. *Stable homotopy*. Lecture Notes in Mathematics, Vol. 165. Springer-Verlag, Berlin, 1970.

[7] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[8] A. D. Elmendorf and M. A. Mandell. Rings, modules, and algebras in infinite loop space theory. *Adv. Math.*, 205(1):163–228, 2006.

[9] Tyler Lawson. *Derived representation theory of nilpotent groups.* ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–Stanford University.

[10] Tyler Lawson. The product formula in unitary deformation K-theory. *K-Theory*, 37(4):395–422, 2006.

[11] Tyler Lawson. The Bott cofiber sequence in deformation K-theory and simultaneous similarity in $\mathrm{U}(n)$. *Math. Proc. Cambridge Philos. Soc.*, 146(2):379–393, 2009.

[12] J. P. May. The spectra associated to permutative categories. *Topology*, 17(3):225–228, 1978.

[13] D. McDuff and G. Segal. Homology fibrations and the "group-completion" theorem. *Invent. Math.*, 31(3):279–284, 1975/76.

[14] Haynes Miller. The Sullivan conjecture on maps from classifying spaces. *Ann. of Math. (2)*, 120(1):39–87, 1984.

[15] Daniel A. Ramras. Excision for deformation K-theory of free products. *Algebr. Geom. Topol.*, 7:2239–2270, 2007.

[16] Daniel A. Ramras. Yang-Mills theory over surfaces and the Atiyah-Segal theorem. *Algebr. Geom. Topol.*, 8(4):2209–2251, 2008.

[17] Daniel A. Ramras. The stable moduli space of flat connections over a surface. *Trans. Amer. Math. Soc.*, 363(2):1061–1100, 2011.

[18] Daniel A. Ramras. Periodicity in the stable representation theory of crystallographic groups. To appear in Forum Math. (published on-line 2012). arXiv:1007.0406, 2012.

[19] Daniel A. Ramras, Rufus Willett, and Guoliang Yu. A finite dimensional approach to the strong Novikov conjecture. arXiv:1203.6168.

[20] Andreas Rosenschon and Paul Arne Østvær. The homotopy limit problem for two-primary algebraic K-theory. *Topology*, 44(6):1159–1179, 2005.

[21] G. B. Segal. Equivariant stable homotopy theory. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 59–63. Gauthier-Villars, Paris, 1971.

[22] Dennis Sullivan. *Geometric topology. Part I.* Massachusetts Institute of Technology, Cambridge, Mass., 1971. Localization, periodicity, and Galois symmetry, Revised version.

[23] R. W. Thomason. The homotopy limit problem. In *Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982)*, volume 19 of *Contemp. Math.*, pages 407–419, Providence, R.I., 1983. Amer. Math. Soc.

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