STABLE REPRESENTATION THEORY FOR CRYSTALLOGRAPHIC GROUPS

DANIEL A. RAMRAS

ABSTRACT. To each discrete group Γ, deformation K–theory associates a spectrum built from spaces of finite dimensional unitary representations of Γ. This provides a stable setting for studying representation spaces. In all known examples, the deformation K–theory spectrum is 2–periodic above the rational cohomological dimension of the group, in the sense that Lawson's Bott map is an isomorphism on homotopy. In this article, we show that a large class of crystallographic groups Γ exhibit such periodicity. This is achieved by showing that for each n, the one-point compactification of the moduli space of irreducible n–dimensional unitary representations of Γ is a CW–complex whose dimension is at most the dimension of the classifying space of Γ. Our methods involve Soren Illman's theory of equivariant triangulations, and the theory of projective representations of finite groups.

As an application of these results, we show that for flat torus bundles E over tori (and some more general Euclidean space forms), the homotopy groups of the stable moduli space of flat, unitary connections vanish above the dimension of E.

1. INTRODUCTION

Associated to a discrete group Γ, one has the unitary representation spaces $\mathrm{Hom}(\Gamma, U(n))$, as well as the moduli spaces $\mathrm{Hom}(\Gamma, U(n))/U(n)$. Deformation K–theory, originally defined by Carlsson, offers a homotopical setting in which to study these spaces. In the author's previous work, the deformation K–theory of a product of surface groups (i.e. fundamental groups of aspherical surfaces) was shown to be 2-periodic above the rational cohomological dimension of the group, minus 2 [14]. Specifically, Lawson's Bott map $\beta\colon \Sigma^2 K^{\mathrm{def}}(\Gamma) \to K^{\mathrm{def}}(\Gamma)$ was shown to induce an isomorphism

$$\pi_*(K^{\mathrm{def}}(\Gamma)) \to \pi_{*+2}(\Gamma)$$

for $* > \mathrm{qcd}(\Gamma) - 2$. In this periodic range, it was also shown that these homotopy groups are isomorphic to the corresponding K–theory groups of $B\Gamma$ (which in this case is a product of surfaces). These results are in precise analogy with the Quillen–Lichtenbaum conjectures in algebraic K–theory, which state that the algebraic K–theory of a scheme should agree with étale K–theory in dimensions greater the (virtual) étale cohomological dimension minus 2. Conjectures of Carlsson [1] (see also the introduction to Lawson [8]) link deformation K–theory to algebraic K–theory of fields, while étale K–theory bears many similarities to ordinary topological K–theory (for example, the Chern character).

These results were used to calculate the homotopy type of the stable moduli space

$$\text{Hom}(\Gamma, U)/U \cong \operatorname*{colim}_n \text{Hom}(\Gamma, U(n))/U(n)$$

when Γ is a product of surface groups. In particular, it was shown that the homotopy groups of this space vanish above the rational cohomological dimension of Γ, and up to torsion these group agree with $H^*(\Gamma; \mathbb{Z})$. Since the classifying space of Γ is a smooth, compact manifold M, this moduli space can also be described as the moduli space $\mathcal{M}_{\text{flat}}(M)$ of flat, unitary connections on M, up to gauge equivalence. Specifically, the holonomy representation associated to a flat bundle $P \to E$ gives rise to a homeomorphism

$$\coprod_P \mathcal{A}_{\text{flat}}(P)/\mathcal{G}(P) \xrightarrow{\cong} \text{Hom}(\Gamma, U(n))/U(n),$$

where the disjoint union is taken over a set of representatives for the isomorphism classes of principal $U(n)$–bundles over M. For further details, we refer the reader to Ramras [13].

The goal of this article is to provide new examples of this sort of periodicity, and to apply them to produce new information about stable moduli spaces of flat connections.

Theorem 1.1. *Let A be a finitely generated abelian group of rank k, and consider an extension*

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1$$

in which Q is a finite abelian group. Then the Bott map

$$\beta\colon K_*^{\text{def}}(\Gamma) \longrightarrow K_{*+2}^{\text{def}}(\Gamma)$$

is an isomorphism for $* > k - 2$.

These groups include, for example, all crystallographic groups with abelian point group (in terms of the above extension, the point group is simply Q). We discuss the relevant background regarding crystallographic groups in Section 2. A standard transfer argument shows that for the groups to which Theorem 1.1 applies, $H^*(\Gamma; \mathbb{Q}) = 0$ for $* > k$, but the rational cohomological dimension of Γ could be less than k. When Γ is crystallographic and torsion-free, Γ acts on $\mathbb{R}^{\text{rk}(A)}$ by isometries, and this is a covering space action [21, Theorem 3.1.3]. The manifold $\mathbb{R}^{\text{rk}(A)}/\Gamma$, which is called a (Euclidean) space form, is then a model for the classifying space $B\Gamma$. If $\mathbb{R}^{\text{rk}(A)}/\Gamma$ is orientable, Γ has (rational) cohomological dimension exactly $\text{rk}(A)$. Thus in this case, our periodicity result fits with the above results on surface groups, and with the Quillen–Lichtenbaum conjectures. It would be extremely interesting if these results could be improved in the case when $\mathbb{R}^k/\Gamma$ is non-orientable, since then the rational cohomological dimension decreases by (at least) one. We note that the case of the Klein bottle, the first non-orientable crystallographic group, is covered by the results of [14], as are all products of Klein bottles and tori. We consider the case of the Klein bottle in Section 10.

When Γ is crystallogrphic but has torsion, then $\mathbb{R}^{\text{rk}(A)}/\Gamma$ is a model for the classifying space for proper actions (usually denoted $\underline{E}\Gamma/\Gamma$). This suggests that for groups with torsion, the periodic range for deformation K-theory may be controlled by the cohomology of $\underline{E}\Gamma/\Gamma$.

In [8, 9], Tyler Lawson constructed spectral sequences linking deformation K–theory to the homology of the one-point compactifications $\text{Irr}_n^+(\Gamma)$ of the moduli

spaces of irreducible $U(n)$–representations of Γ. As we will see in Section 8, Lawson's results allow us to deduce periodicity from the vanishing of $H_*(\mathrm{Irr}_n^+(\Gamma); \mathbb{Z})$ for $* > k$. We will prove the following stronger result.

Theorem 1.2. *If Γ sits in an extension*

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow Q \longrightarrow 1$$

as in Theorem 1.1, then each space $\mathrm{Irr}_n^+(\Gamma)$ is homeomorphic to a CW–complex of dimension at most $\mathrm{rk}(A)$.

To produce the necessary CW–structures, we will use results of Soren Illman regarding equivariant triangulations of algebraic varieties. The bound on dimensions will come from combining information about the induction maps

$$\mathrm{Ind}_H^\Gamma\colon \mathrm{Hom}(H, U(n)) \longrightarrow \mathrm{Hom}(\Gamma, U(n))$$

for $A \leqslant H < \Gamma$ with information about projective representations of the quotient group Q. We note that explicit computations for two particular crystallographic groups, of the form $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$, were given by Lawson [9]. In these cases one sees 1– and 2–dimensional moduli spaces of irreducibles (respectively), and these are the maximum possible dimensions allowed by Theorem 1.2.

As an application of our results we show that for every flat torus bundle E over a torus, the homotopy groups of the stable moduli space

$$\mathrm{Hom}(\pi_1 E, U)/U \cong \operatorname*{colim}_n \mathrm{Hom}(\pi_1 E, U(n))/U(n)$$

vanish above the dimension of E. These moduli spaces may be viewed as stable moduli space of flat, unitary connections over E (note here that the term *flat* is being used in two somewhat different ways). As explained in Section 9.2, this depends on work of Ratcliffe and Tschantz [16], who showed that $\pi_1 E$ has a particularly nice form. In fact, this result can be iteratively strengthened by replacing the base torus by a flat torus bundle over a torus (and so on), although we must still require that the point group of $\pi_1 E$ is abelian. Without assuming the point group is abelian, we can still prove *rational* periodicity (Proposition 9.7); this is easier and is essentially a variant on standard transfer-type arguments.

This paper is structured as follows. In Section 3, we introduce notation for the various spaces of representations we will be considering, and establish some basic facts about their topology. In Section 4, we explain a basic fact from representation theory, which guides our arguments by dividing irreducible representations into two classes: induced representations, and those which produce projection representations of the quotient group Q. In Sections 5 and 6, we study these two classes of representations. Section 7 establishes our results on the dimension of the moduli space of irreducible representations, and Section 8 explains the application to deformation K–theory. In Section 9.2, we discuss the application to stable moduli spaces of torus bundles over tori. In the final section, we describe an explicit computation for the Klein bottle.

2. Crystallographic groups

A crystallographic group is a discrete subgroup Γ of the isometries of Euclidean space $\mathbb{R}^k$, such that $\mathbb{R}^k/\Gamma$ is compact (we call k the *dimension* of Γ). Standard references for the theory of cyrstallographic groups are Ratcliffe [15, Section 7.5] and Wolf [21, Chapter 3].

There are finitely many crystallographic groups in each dimension, and from the classification in Wolf [21, Theorems 3.5.9 and 3.5.10], one sees, for example, that all three-dimensional crystallographic groups have abelian point group. In dimensions 4, 5, and 6, non-abelian point groups appear, although there are many crystallographic groups in these dimensions with abelian point group (see, for example, [2]).

By [15, Theorem 7.5.2] and [21, Theorem 3.2.9], every crystallographic group Γ of dimension k contains a free abelian normal subroup $A \cong \mathbb{Z}^k$ (the subgroup of *translations*) which is a maximal abelian subgroup of Γ. The *point group* of Γ is then the quotient Γ/A. Hence all cystallographic groups sit in extensions of the form $A \hookrightarrow \Gamma \twoheadrightarrow Q$ with A *free* abelian and Q finite. Note that in any such extension, conjugation in Γ induces an action of Q on A. It turns out (see [15, Theorem 7.5.5]) that crystallographic groups are precisely those groups Γ sitting in an extension $\mathbb{Z}^k \to \Gamma \to Q$, with Q a finite group acting on A via an embedding $Q \hookrightarrow \mathrm{GL}_k(\mathbb{Z}) = \mathrm{Aut}(A)$.

Since we will be working with abstract groups, rather than with groups equipped with a specific action on Euclidean space, it is helpful to know that the translation subgroup and the point group can be defined abstractly, independent of any chosen action. For completeness, we give a proof. A similar result is proven in Wolf [21, Theorem 3.2.9].

Lemma 2.1. *Say Γ is isomorphic to a crystallographic group of isometries of Euclidean space. Then there is exactly one subgroup of $A \leqslant \Gamma$ satisfying the following three conditions: A is free abelian, of finite index, and maximal abelian in Γ. This subgroup is normal in Γ, and it follows that under any isomorphism of Γ with a crystallographic subgroup of the isometries of Euclidean space, A is mapped onto the subgroup of translations. We call A the translation subgroup of Γ.*

Proof. If A and A' are two finite index, free abelian subgroups of a group G, then the free abelian group $A \cap A'$ must have finite index in both A and A', so $\mathrm{rk}(A) = \mathrm{rk}(A \cap A') = \mathrm{rk}(A')$.

Now, if Γ is isomorphic to a crystallographic group, then let $A \leqslant \Gamma$ be the subgroup of translations determined by this isomorphism. Then A has all the desired properties. If $A' \leqslant \Gamma$ is another free abelian subgroup of finite index, which is also maximal abelian in Γ, then we know $\mathrm{rk}(A') = \mathrm{rk}(A)$, and A' acts on $\mathbb{R}^{\mathrm{rk}(A)}$ as a discrete group of isometries. By Ratcliffe [15, Theorem 5.4.4], any free abelian discrete subgroup of rank k in $\mathrm{Isom}(\mathbb{R}^n)$ acts by translations on some k–plane; since in our case A' has full rank, we see that it acts by translations on all of $\mathbb{R}^{\mathrm{rk}(A)}$. Now, [15, Theorem 5.4.3] states that there exists an abelian normal subgroup $N \triangleleft \Gamma$ containing all the translations, so both A and A' must lie in N. But A and A' are maximal abelian, so $A = A' = N$. $\square$

3. Spaces of representations

Let Γ be an infinite discrete group. In this section we set up some basic terminology and record some facts about the point-set topology of spaces of unitary representations of Γ. We denote the set of homomorphisms $\rho\colon \Gamma \to U(n)$ by $\mathrm{Hom}(\Gamma, U(n))$. This space is naturally topologized using product topology on $U(n)^\Gamma$ (which is the same as the compact-open topology on $\mathrm{Map}(\Gamma, U(n)) \cong U(n)^\Gamma$). One may check that $\mathrm{Hom}(\Gamma, U(n))$ is closed in $U(n)^\Gamma$, hence compact. A generating set $S \subset \Gamma$ determines an injection from $\mathrm{Hom}(\Gamma, U(n))$ into a product of $|S|$ copies of $U(n)$, and by compactness this map is always a homeomorphism onto its image. If our generating set contains $m < \infty$ elements, then the corresponding embedding gives $\mathrm{Hom}(\Gamma, U(n))$ the structure of a real algebraic variety, cut out from the real algebraic variety $U(n)^m$ by the relations in Γ. For our purposes, it will not be necessary to consider the relationship between the algebraic structures induced by different generating sets.

The block sum maps $U(n) \times U(m) \to U(n+m)$, which we denote $(A, B) \mapsto A \oplus B$, determine corresponding block sum maps on representation spaces, which we again denote $(\rho, \psi) \mapsto \rho \oplus \psi$. The action of the unitary group on itself by conjugation induces an action of $U(n)$ on $\mathrm{Hom}(\Gamma, U(n))$.

Definition 3.1. *The* moduli space of $U(n)$–representations *is the quotient space* $\mathrm{Hom}(\Gamma, U(n))/U(n)$.

Let $\mathrm{Irr}(\Gamma, U(n)) \subset \mathrm{Hom}(\Gamma, U(n))$ *denote the subspace of irreducible representations, i.e. those which are not isomorphic to $\rho \oplus \psi$ for any ρ, ψ. Let* $\mathrm{Sum}(\Gamma, U(n)) \subset \mathrm{Hom}(\Gamma, U(n))$ *denote the complement of* $\mathrm{Irr}(\Gamma, U(n))$. *The* moduli space of irreducible $U(n)$–representations *is the quotient space* $\mathrm{Irr}_n(\Gamma) = \mathrm{Irr}(\Gamma, U(n))/U(n)$.

Lemma 3.2. *For any discrete group Γ, the space* $\mathrm{Sum}(\Gamma, U(n)) \subset \mathrm{Hom}(\Gamma, U(n))$ *is closed.*

Proof. The block sum maps descend to continuous maps

(1) $\mathrm{Hom}(\Gamma, U(k)/U(k) \times \mathrm{Hom}(\Gamma, U(n-k))/U(n-k) \longrightarrow \mathrm{Hom}(\Gamma, U(n))/U(n),$

and $\mathrm{Sum}(\Gamma, U(n))/U(n)$ is the union of the images of the maps (1) as k ranges from 1 to $n-1$. Since the domains of these maps are compact, $\mathrm{Sum}(\Gamma, U(n))/U(n)$ is closed, and hence so is its inverse image $\mathrm{Sum}(\Gamma, U(n))$ in $\mathrm{Hom}(\Gamma, U(n))$. $\qquad \square$

Note that $\mathrm{Irr}_n(\Gamma)$ is the complement of $\mathrm{Sum}(\Gamma, U(n))/U(n)$ inside the moduli space $\mathrm{Hom}(\Gamma, U(n))/U(n)$. Lawson's articles [8] and [9] use different notations for the space

$$\big(\mathrm{Hom}(\Gamma, U(n))/U(n)\big)/\big(\mathrm{Sum}(\Gamma, U(n))/U(n)\big),$$

which contains the moduli space of irreducibles as a subspace (the complement of the basepoint). The following observation motivates a more natural name for this space.

Lemma 3.3. *The space* $\mathrm{Hom}(\Gamma, U(n))/U(n)$ *is compact Hausdorff, and*

$$\mathrm{Irr}_n^+(\Gamma)\colon \; = \big(\mathrm{Hom}(\Gamma, U(n))/U(n)\big)/\big(\mathrm{Sum}(\Gamma, U(n))/U(n)\big)$$
$$\cong \big(\mathrm{Hom}(\Gamma, U(n))/\mathrm{Sum}(\Gamma, U(n))\big)/U(n)$$

is the one-point compactification of the moduli space of irreducible representations.

Proof. By Lemma 3.2, $\mathrm{Sum}(\Gamma, U(n))$ is a closed subspace of $\mathrm{Hom}(\Gamma, U(n))$. Since $\mathrm{Hom}(\Gamma, U(n))$ is regular, the quotient $\mathrm{Hom}(\Gamma, U(n))/\mathrm{Sum}(\Gamma, U(n))$ is Hausdorff. In general, the quotient of a Hausdorff space by a compact group is Hausdorff (see for example Munkres [10, Exercise 31.8]), so the action of $U(n)$ on either $\mathrm{Hom}(\Gamma, U(n))$ or $\mathrm{Hom}(\Gamma, U(n))/\mathrm{Sum}(\Gamma, U(n))$ produces a Hausdorff quotient space. This proves the first statement. For the second statement, we now know that the space $\mathrm{Irr}_n^+(\Gamma)$ is a compact Hausdorff space, and the moduli space of irreducibles embeds in $\mathrm{Irr}_n^+(\Gamma)$ as the complement of the basepoint corresponding to $\mathrm{Sum}(\Gamma, n)/U(n)$. Every compact Hausdorff space X is the one-point compactification of $X - \{x\}$ for each $x \in X$ (see, for example, Munkres [10, Theorem 29.1]), so the result follows. $\square$

We will denote that point at infinity in $\mathrm{Irr}_n^+(\Gamma)$ by $+$, to emphasize the fact that it corresponds to the space of reducible representations.

Remark 3.4. *All spaces encountered in this paper are Hausdorff, as can be shown using the same methods as in Lemma 3.3.*

Although we will not need it for the main results of this paper, we note a simple fact which is extremely helpful in computations.

Proposition 3.5. *If Γ contains an abelian subgroup A of finite index m, then every irreducible unitary representation of Γ is at most m–dimensional. Hence $\mathrm{Irr}_n^+(\Gamma) = \{+\}$ for $n > m$.*

This is proven in Serre [19, Section 3.1]. The essential point is that simultaneously commuting matrices are simultaneously diagonalizable, so an irreducible representation of A is 1–dimensional. The result follows by restricting an irreducible representation $\rho : \Gamma \to U(n)$ to a representation ρ_A of A, and noting that the translates under G of any irreducible summand in ρ_A generate $\mathbb{C}^n$.

4. Induced Representations and Projective Representations

The overall structure of our arguments is based on the following result from representation theory (see Serre [19, Proposition 24]).

Theorem 4.1. *Let A be an abelian normal subgroup of finite index in a discrete group Γ. For every irreducible representation $\rho\colon \Gamma \to U(n)$, either*

- *ρ is isomorphic to $\mathrm{Ind}_H^\Gamma(\rho')$ for some proper subgroup $H < \Gamma$ containing A and some irreducible unitary representation ρ' of H, or*
- *the restriction of ρ to A is scalar (a direct sum of isomorphic 1–dimensional representations).*

This result is proven by letting H be the stabilizer of an isotypic component of ρ. Serre's book focuses on finite groups, but the proof extends without change to (unitary) representations of infinite discrete groups. Here we use the fact that all irreducible unitary representations of A are 1-dimensional. (This follows from the fact that commuting unitary matrices are simultaneously diagonalizable.)

Theorem 4.1 shows that if Γ sits in an extension

$$A \longrightarrow \Gamma \longrightarrow Q$$

with A finitely generated abelian and Q finite abelian, then every representation $\rho\colon \Gamma \to U(n)$ is either induced from a some $H < \Gamma$ with $A \leqslant H$, or else induces a homomorphism $\bar{\rho}\colon Q = \Gamma/A \to \mathrm{PU}(n)$, where $\mathrm{PU}(n)$, the projective unitary group, denotes the quotient of $U(n)$ by the subgroup of scalar matrices λI, $\lambda \in S^1$. (From now on, we will denote this subgroup simply by $S^1 \subset U(n)$.) In the former case, the group H sits in an extension

$$A \longrightarrow H \longrightarrow H/A,$$

and $H/A \leqslant Q$ is abelian. Thus H still satisfies the hypotheses of our theorems, and this will allow for an induction argument based on the order of the quotient group Q. We will analyze these two classes of representations (induced and projective) separately in the following sections, and then combine our results to prove the main theorems.

5. INDUCTION

In this section, Γ will denote a discrete group with a normal subgroup $H \lhd \Gamma$ of index $m < \infty$. In order to use Theorem 4.1, we need to analyze the induction maps

$$\mathrm{Ind}_H^\Gamma\colon \ \mathrm{Hom}(H, U(n))/U(n) \longrightarrow \mathrm{Hom}(\Gamma, U(nm))/U(nm).$$

Abstractly, induction can be defined by viewing a representation $\rho\colon H \to U(n)$ as a (left) module V over the group ring $\mathbb{C}[H]$, and defining $\mathrm{Ind}_H^\Gamma(V)$ to be the (left) $\mathbb{C}[\Gamma]$ module $\mathbb{C}[\Gamma] \otimes_{\mathbb{C}[H]} V$.

We need a description of induction as a continuous function between spaces of unitary matrices. This is easily obtained by choosing a set of coset representatives $\gamma_1, \ldots, \gamma_m$ for Γ/H, which yields a direct sum decomposition

$$\mathbb{C}[\Gamma] \cong \gamma_1 \mathbb{C}[H] \oplus \cdots \oplus \gamma_m \mathbb{C}[H]$$

of $\mathbb{C}[\Gamma]$ as a *right* $\mathbb{C}[H]$–module. Now, a representation $\rho\colon H \to U(n)$ gives $\mathbb{C}^n$ a left $\mathbb{C}[H]$–module structure, and we denote this module by $\mathbb{C}_\rho^n$. We now have a canonical ordered basis for the complex vector space $\mathbb{C}[\Gamma] \otimes_{\mathbb{C}[H]} \mathbb{C}_\rho^n$, given by

$$\gamma_1 \otimes e_1, \gamma_1 \otimes e_2, \cdots, \gamma_1 \otimes e_n, \gamma_2 \otimes e_1, \cdots, \gamma_2 \otimes e_n, \cdots, \gamma_m \otimes e_1, \cdots, \gamma_m \otimes e_n.$$

We endow $\mathbb{C}[\Gamma] \otimes_{\mathbb{C}[H]} \mathbb{C}_\rho^n$ with a Hermitian metric by declaring the above basis orthonormal, and we claim that the action of each $\gamma \in \Gamma$ on $\mathbb{C}[\Gamma] \otimes_{\mathbb{C}[H]} \mathbb{C}_\rho^n$ is now unitary. Indeed, if $\gamma\gamma_i = \gamma_j h$, then

$$\gamma \cdot (\gamma_i \otimes e_k) = \gamma_j h \otimes e_k = \gamma_j \otimes \rho(h) e_k = \sum_l \rho(h)_{lk}(\gamma_j \otimes e_l),$$

where $\rho(h)_{lk}$ is the entry $(l,k)^{\mathrm{th}}$ entry (l^{th} row, k^{th} column) of $\rho(h) \in U(n)$. If we write out the action of γ as a matrix in our chosen basis, we see that the column corresponding to $\gamma_i \otimes e_k$ contains one copy of the kth column of $\rho(h)$, with zeros elsewhere. Thus each column has length 1 (since $\rho(h) \in U(n)$). Moreover, since each $\gamma \in \Gamma$ permutes the cosets $\{\gamma_i H\}_{i=1}^m$, we see that the columns of this matrix are orthogonal: no row contains non-zero elements in the columns for $\gamma_i \otimes e_k$ and $\gamma_j \otimes e_k$ (if $i \neq j$), and the columns for $\gamma_i \otimes e_k$ and $\gamma_i \otimes e_l$ consist of the the kth and l^{th} columns of $\rho(h)$ (which are orthogonal) placed in rows corresponding to $\gamma_j \otimes e_1, \ldots, \gamma_j \otimes e_n$.

Thus we have a well-defined induction map

$$(2) \qquad \operatorname{Ind}_H^\Gamma \colon \operatorname{Hom}(H, U(n)) \longrightarrow \operatorname{Hom}(\Gamma, U(n)),$$

and it is continuous because the $(i,j)^{\text{th}}$ entry of the matrix representing $\operatorname{Ind}_H^\Gamma(\rho)(\gamma)$ is simply an entry of $\rho(h)$, for some h depending only on γ, i, and j. Moreover, we claim that this map descends to a continuous map

$$(3) \qquad \operatorname{Ind}_H^\Gamma \colon \operatorname{Hom}(H, U(n))/U(n) \longrightarrow \operatorname{Hom}(\Gamma, U(nm))/U(nm).$$

Indeed, if $\rho \cong \rho'$, then $\mathbb{C}_\rho^n \cong \mathbb{C}_{\rho'}^n$ as $\mathbb{C}[H]$–modules, so $\mathbb{C}[\Gamma] \otimes_{\mathbb{C}[H]} \mathbb{C}_\rho^n \cong \mathbb{C}[\Gamma] \otimes_{\mathbb{C}[H]} \mathbb{C}_{\rho'}^n$ as $\mathbb{C}[G]$–modules. Therefore, $P\big(\operatorname{Ind}_H^\Gamma(\rho)\big)P^{-1} = \operatorname{Ind}_H^\Gamma(\rho')$ for some $P \in \operatorname{GL}_n(\mathbb{C})$. It follows from Proposition 6.5 (below) that if two unitary representations of a group Γ are conjugate as $\operatorname{GL}_n(\mathbb{C})$–representations, then they are conjugate as $U(n)$–representations, but in this case one can see explicitly that if $\rho' = X\rho X^{-1}$, then in the above basis, one has $\operatorname{Ind}_H^\Gamma(\rho') = (mX)\big(\operatorname{Ind}_H^\Gamma(\rho)\big)(mX)^{-1}$, where mX denotes the block sum of m copies of the matrix X. Hence if $X \in U(n)$, so is mX.

We note that although the maps (2) certainly depend on our chosen isomorphism $\mathbb{C}[\Gamma] \otimes_{\mathbb{C}[H]} \mathbb{C}^n \cong \mathbb{C}^{nm}$, the maps (3) are independent of this choice. If we choose another isomorphism with $\mathbb{C}^{nm}$ such that the corresponding representations $(\operatorname{Ind}_H^\Gamma(\rho))'$ still lie in $U(nm)$, then $\operatorname{Ind}_H^\Gamma(\rho)$ and $(\operatorname{Ind}_H^\Gamma(\rho))'$ are unitary representations that are linearly isomorphic, and hence they are linearly isometric as well (see Corollary 6.6, for example).

Having established continuity of the induction maps, we now consider some of their basic properties. First, note that if ρ is not irreducible, then neither is $\operatorname{Ind}_H^\Gamma(\rho)$. Hence we obtain induced maps, which we will still denote by $\operatorname{Ind}_H^\Gamma$,

$$\operatorname{Ind}_H^\Gamma \colon \operatorname{Irr}_n^+(H) \to \operatorname{Irr}_{nm}^+(\Gamma).$$

We now discuss Frobenius Reciprocity in the context of unitary representations of an infinite discrete group Γ. Given unitary representations ψ and ψ' of Γ, we may consider ψ and ψ' as vector spaces with an action of Γ, and we denote the dimension of the vector space of Γ–equivariant linear maps $\psi \to \psi'$ by

$$\langle \psi, \psi' \rangle_\Gamma.$$

Frobenius reciprocity states that if $H < \Gamma$ has finite index, ψ is a representation of H, and ϕ is a representation of Γ, then

$$\langle \psi, \operatorname{Res}_H^\Gamma \phi \rangle_H = \langle \operatorname{Ind}_H^\Gamma \psi, \phi \rangle_\Gamma.$$

The proof given by Serre [19, Section 7.2]] goes through in this context. This makes use of the following basic lemma, which results from Schur's Lemma (automorphisms of irreducibles are scalar) and the fact that unitary representations always admit decompositions into irreducible direct summands, unique up to isomorphism and permutation of the summands.

Lemma 5.1. *If ρ is a unitary representation of Γ, then ρ is irreducible if and only if $\langle \rho, \rho \rangle = 1$.*

Frobenius Reciprocity allows us to identify irreducible induced representations.

Lemma 5.2. *There is an action of Γ/H on $\operatorname{Irr}_n^+(H)$, and $\operatorname{Ind}_H^\Gamma(\rho)$ is irreducible if and only if Γ/H acts freely on the orbit of $[\rho]$.*

Proof. There is an action of Γ on $\mathrm{Hom}(H, U(n))$, given by $(\gamma \cdot \rho)(h) = \rho(\gamma h \gamma^{-1})$. This action descends to a well-defined action on $U(n)$–isomorphism classes. Each $h_0 \in H$ acts trivially because $\rho(h_0) \in U(n)$ provides an isomorphism between ρ and $h_0 \cdot \rho$, so we in fact have an action of Γ/H. This action preserves the space of reducible representations, so it induces the desired action on $\mathrm{Irr}_n^+(H)$.

Now, Frobenius Reciprocity states that

$$\langle \mathrm{Ind}_H^\Gamma(\rho), \mathrm{Ind}_H^\Gamma(\rho) \rangle_\Gamma = \langle \rho, \mathrm{Res}_H^\Gamma \mathrm{Ind}_H^\Gamma \rho \rangle.$$

Since H is normal in Γ, we have

$$\langle \rho, \mathrm{Res}_H^\Gamma \mathrm{Ind}_H^\Gamma \rho \rangle = \langle \rho, \bigoplus_{[\gamma] \in \Gamma/H} \gamma \cdot \rho \rangle$$

(this follows from [19, Section 7.3, Proposition 22], for example). The right-hand side is equal to 1 if and only if the action of Γ/H on the orbit of $[\rho]$ is free, since otherwise ρ appears at least twice (up to isomorphism) in the sum on the right. Lemma 5.1 completes the proof. $\square$

Definition 5.3. *Let* $\mathrm{Irr}_n(H)^{\mathrm{free}}$ *denote the subspace of* $\mathrm{Irr}_n^+(H)$ *on which* Γ/H *acts freely, and let* $\mathrm{Irr}_{nm}(\Gamma)_H$ *denote the subspace of* $\mathrm{Irr}_{nm}^+(\Gamma)$ *consisting of those irreducible representations induced from* n–*dimensional (irreducible) representations of* H. *Note that* $\mathrm{Irr}_n(H)^{\mathrm{free}}$ *and* $\mathrm{Irr}_{nm}(\Gamma)_H$ *are homeomorphic to subspaces of* $\mathrm{Hom}(H, U(n))/U(n)$ *and* $\mathrm{Hom}(\Gamma, U(nm))/U(nm)$, *respectively.*

Lemma 5.4. *The subspace* $\mathrm{Irr}_n(H)^{\mathrm{free}}$ *is open in* $\mathrm{Hom}(H, U(n))/U(n)$.

Proof. In general, if X is Hausdorff and G is a finite group acting on X, then the subspace on which G acts freely is open. Indeed, its complement can be written as

$$\bigcup_{g \in G} \{ x \in X \mid g \cdot x = x \},$$

and each of these sets is closed because it is the inverse image of the diagonal under the continuous map $X \to X \times X$, $x \mapsto (x, g \cdot x)$. $\square$

The quotient map for a free action of a finite groups is a covering map, so we obtain the following result, which plays a key role in Section 7.

Proposition 5.5. *The induction map* Ind_H^Γ *induces a degree* $|\Gamma/H|$ *covering map*

$$\mathrm{Irr}_n(H)^{\mathrm{free}} \twoheadrightarrow \mathrm{Irr}_{nm}^+(\Gamma)_H.$$

In Section 9.2, we will need to consider the interaction between induction and tensor products. By a construction similar to that given above for induction maps, one obtains continuous maps

$$\mathrm{Hom}(\Gamma, U(n)) \times \mathrm{Hom}(\Gamma, U(m)) \xrightarrow{\otimes} \mathrm{Hom}(\Gamma, U(nm)),$$

which depend on a choice of ordered basis for the vector space $\mathbb{C}^n \otimes \mathbb{C}^m$, and which descend to continuous maps on the moduli spaces. These latter maps are independent of the choices made, and in terms of $\mathbb{C}[G]$–modules, they send a pair of modules V and W to the module $V \otimes_{\mathbb{C}} W$, with the diagonal action of $\mathbb{C}[G]$.

The following well-known fact (see, for example, Serre [19, §7.2]) is sometimes called the Projection Formula.

Lemma 5.6. *For any representations* $\rho \colon \Gamma \to U(n)$ *and* $\psi \colon H \to U(k)$, *we have*

$$\mathrm{Ind}_H^\Gamma \left((\mathrm{Res}_H^\Gamma \rho) \otimes \psi \right) \cong \rho \otimes \mathrm{Ind}_H^\Gamma(\psi).$$

6. Projective representations

In this section, Γ will denote a discrete group, $A \lhd \Gamma$ will denote a finite index normal subgroup, and $Q = \Gamma/A$ will denote the quotient group.

Recall from Section 4 that if A is abelian, each irreducible representation $\rho \colon \Gamma \to U(n)$ is either induced from a subgroup containing A, or satisfies $\rho(A) \subset S^1$. In the latter case, we obtain a representation $\bar{\rho} \colon Q = \Gamma/A \to \mathrm{PU}(n) = U(n)/S^1$ (which we will avoid referring to as the "induced representation"...).

Definition 6.1. *Let* $\mathrm{Hom}_A(\Gamma, U(n))$ *denote the subspace of* $\mathrm{Hom}(\Gamma, U(n))$ *consisting of representations* ρ *such that* $\rho(A) \subset S^1$.

Lemma 6.2. *If* Γ *is finitely generated, then* $\mathrm{Hom}_A(\Gamma, U(n))$ *is a closed,* $U(n)$–*invariant subvariety of* $\mathrm{Hom}(\Gamma, U(n))$. *In particular,* $\mathrm{Hom}_A(\Gamma, U(n))$ *is compact.*

Proof. Invariance under $U(n)$ follows from the fact that S^1 is central in $U(n)$. The subspace $\mathrm{Hom}_A(\Gamma, U(n))$ is cut out from $\mathrm{Hom}(\Gamma, U(n))$ by the requirements that for each $a \in A$, all the off-diagonal entries of $\rho(a)$ are zero and all of the diagonal entries are equal to one another. Thus $\mathrm{Hom}_A(\Gamma, U(n))$ is the set of common zeros of a collection of polynomials, and hence is a closed subvariety of $\mathrm{Hom}(\Gamma, U(n))$. $\qquad \square$

Given a discrete group G, the space $\mathrm{Hom}(G, \mathrm{PU}(n))$ has the subspace topology from the product space $\mathrm{PU}(n)^G$. This has a natural action of $\mathrm{PU}(n)$ by conjugation. Using Schur's theory of projective representations of finite groups, we will be able to analyze the irreducible representations in $\mathrm{Hom}_A(\Gamma, U(n))$ in terms of the map $\mathrm{Hom}_A(\Gamma, U(n)) \to \mathrm{Hom}(Q, \mathrm{PU}(n))$. We will now set up some basic terminology regarding projective representations. For more detail about this subject, we refer the reader to Karpilovsky [7, Chapter 3].

Homomorphisms $G \to \mathrm{PU}(n)$ are closely connected to what are usually called projective unitary representations, that is, functions $\rho \colon G \to U(n)$ such that $\rho(g_1)\rho(g_2) = \sigma(g_1, g_2)\rho(g_1 g_2)$ for some scalar $\sigma(g_1, g_2) \in S^1$ (we assume that $\rho(1) = 1$). Replacing $U(n)$ by $\mathrm{GL}_n(\mathbb{C})$ and S^1 by $\mathbb{C}^*$ yields the notion of a projective linear representation. Each homomorphism $G \to \mathrm{PU}(n)$ may be lifted (in many ways) to a projective unitary representation, and conversely each projective representation defines a homomorphism $G \to \mathrm{PU}(n)$.

There are four notions of equivalence between projective unitary representations $\rho \colon G \to U(n)$ and $\rho' \colon G \to U(n)$. We say that ρ and ρ' are *projectively equivalent* if there exists a matrix $A \in \mathrm{GL}_n(\mathbb{C})$ such that for each $g \in G$, $A\rho(g)A^{-1}\rho'(g)^{-1} \in \mathbb{C}^*$. We write $\rho \sim_{\mathrm{GL}} \rho'$ in this situation. By requiring the matrix A to lie in $U(n)$, we obtain a (potentially) stricter notion of equivalence, *projective unitary equivalence*, which we denote by $\rho \sim_U \rho'$. Next, we say that ρ and ρ' are *linearly equivalent* ($\rho \approx_{\mathrm{GL}} \rho'$) if there exists a matrix $A \in \mathrm{GL}_n(\mathbb{C})$ such that $A\rho(g)A^{-1} = \rho'(g)$ for all $g \in G$. Finally, we obtain the notion of *unitary linear equivalence* ($\rho \approx_U \rho'$) by requiring the matrix A to lie in $U(n)$.

The reader may check that isomorphism classes (i.e. $\mathrm{PU}(n)$ conjugacy classes) of homomorphisms $G \to \mathrm{PU}(n)$ correspond bijectively with *projective unitary equivalence classes* of projective representations.

We say that a projective representation $\rho \colon G \to U(n)$ is *irreducible* if there is no proper, non-zero subspace of $\mathbb{C}^n$ that is invariant under each of the matrices $\rho(g)$, $g \in G$. We say that a homomorphism $G \to \mathrm{PU}(n)$ is irreducible if one of its

lifts to a projective representation is irreducible. Note that if ρ and ρ' are two such lifts, then for each $g \in G$, $\rho(g)$ and $\rho'(g)$ differ by a scalar, and hence the invariant subspaces for ρ and ρ' are the same. Thus *all* projective representations associated to an irreducible homomorphism $G \to \mathrm{PU}(n)$ are irreducible. On the other hand, if $\rho\colon G \to U(n)$ is an irreducible projective representation, then the corresponding homomorphism $\overline{\rho}\colon G \to \mathrm{PU}(n)$ is irreducible as well, since ρ itself is a lift of $\overline{\rho}$.

We note that irreducibility is preserved under projective equivalence: if $\rho\colon G \to U(n)$ is irreducible and $\rho \sim_{\mathrm{GL}} \rho'$, then ρ' is irreducible as well. In particular, a homomorphism $\rho\colon G \to \mathrm{PU}(n)$ is irreducible if and only if all of its $\mathrm{PU}(n)$–conjugates are irreducible.

We note another simple but important fact regarding irreducible projective representations.

Lemma 6.3. *A representation $\rho \in \mathrm{Hom}_A(\Gamma, U(n))$ is irreducible if and only if $\overline{\rho}\colon Q \to \mathrm{PU}(n)$ is irreducible.*

Proof. Choose a set of coset representatives $\{\gamma_i\}_i$ for the cosets of A, and lift $\overline{\rho}$ to a projective representation $\widetilde{\rho}\colon Q \to U(n)$ by setting $\widetilde{\rho}([\gamma_i]) = \rho(\gamma_i)$. Note that for each $\gamma \in \Gamma$, we have $\gamma = \gamma_i a$ for some $a \in A$ and some i, so $\rho(\gamma) = \rho(\gamma_i a) = \widetilde{\rho}([\gamma_i])\rho(a)$. Since $\rho(A) \subset S^1$, we see that each matrix comprising the representation ρ is a scalar multiple of a matrix appearing in $\widetilde{\rho}$, and of course each matrix appearing in $\widetilde{\rho}$ also appears in ρ. Hence $\widetilde{\rho}$ and ρ have the same invariant subspaces, so $\widetilde{\rho}$ is irreducible if and only if ρ is irreducible. $\square$

Our next goal is to show that for irreducible projective unitary representations $\rho\colon G \to U(n)$, the equivalence relations $\approx_{\mathrm{GL}}$ and $\approx_U$ coincide, as do the relations $\sim_U$ and $\sim_{\mathrm{GL}}$.

Lemma 6.4. *Let $\rho\colon G \to \mathrm{GL}(V)$ be an irreducible projective representation of G on a finite-dimensional complex vector space V. If $\langle\, ,\, \rangle$ and $\langle\, ,\, \rangle'$ are two ρ–invariant Hermitian metrics on V, (i.e. $\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle$ for all $g \in G$, and similarly for $\langle\, ,\, \rangle'$) then there exists a scalar $t \in \mathbb{R}^+$ such that*

$$\langle v, w \rangle' = t \langle v, w \rangle$$

for all $v, w \in V$.

Proof. Let V^* denote the dual space consisting of *conjugate*–linear functionals on V; that is

$$V^* = \{ f\colon V \to \mathbb{C} \mid \forall\, v, w \in V,\ \lambda \in \mathbb{C},\ f(v + \lambda w) = f(v) + \overline{\lambda} f(w) \}.$$

Note that V^* is a complex vector space under point-wise addition and (ordinary) scalar multiplication. Any Hermitian metric $\langle\, ,\, \rangle$ on V induces a complex-linear isomorphism $\phi\colon V \xrightarrow{\cong} V^*$, where $\phi(v) = \phi_v\colon V \to \mathbb{C}$ is given by the formula

$$\phi_v(w) = \langle v, w \rangle \in \mathbb{C}.$$

Now, any projective representation $\rho\colon G \to \mathrm{GL}(V)$ induces a projective representation $\rho^*\colon G \to \mathrm{GL}(V^*)$ (called the *contragradient* of ρ). If we write the resulting actions of G on V and V^* as $\rho(g)v = g \cdot v$ and $\rho^*(g)f = g \cdot f$, then ρ^* is defined by the formula

$$(g \cdot f)(v) = f(g^{-1} \cdot v).$$

One may now check that $g \cdot f$ is conjugate linear, and that, up to multiplication by scalars, this formula gives an action of G on V^*. Hence ρ^* is a projective representation of G.

Next, say we have two ρ–invariant metrics $\langle\,,\,\rangle$ and $\langle\,,\,\rangle'$ on V. Then a short computation shows that the resulting isomorphisms $\phi, \phi'\colon V \to V^*$ are G–equivariant, with respect to the representations ρ and ρ^*. Hence the linear isomorphism

$$X = \phi^{-1}\phi'\colon V \to V$$

satisfies $X^{-1}\rho X = \rho$. Since ρ is irreducible, the usual proof of Schur's lemma shows that $X = tI$ for some $t \in \mathbb{C}$. Now for any $v \in V$, $\phi'(v) = \phi(tv) = t\phi(v)$, meaning that

$$\langle v, w\rangle' = t\langle v, w\rangle$$

for any $w \in V$. All that remains is to check that $t \in \mathbb{R}^+$; this follows from the assumption that $\langle\,,\,\rangle$ and $\langle\,,\,\rangle'$ are (positive definite) Hermitian metrics. $\qquad\square$

Proposition 6.5. *If $\rho, \rho'\colon G \to U(n)$ are irreducible projective unitary representations of a discrete group G, then $\rho \approx_{\mathrm{GL}} \rho' \iff \rho \approx_U \rho'$.*

Proof. The direction $\rho \approx_U \rho' \implies \rho \approx_{\mathrm{GL}} \rho'$ is immediate. For the converse, let $P \in \mathrm{GL}_n(\mathbb{C})$ be a matrix satisfying $P^{-1}\rho(g)P = \rho'(g)$ for all $g \in G$. Define a Hermitian metric on $\mathbb{C}^n$ by setting $\langle v, w\rangle_P = \langle Pv, Pw\rangle$, where $\langle\,,\,\rangle$ is the standard Hermitian metric on $\mathbb{C}^n$. Then $\langle\,,\,\rangle_P$ is an invariant metric for the representation ρ', because

$$\langle \rho'(g)v, \rho'(g)w\rangle_P = \langle P\rho'(g)v, P\rho'(g)w\rangle = \langle \rho(g)Pv, \rho(g)Pw\rangle$$
$$= \langle Pv, Pw\rangle = \langle v, w\rangle_P,$$

with the third equality following from the fact that $\rho(g) \in U(n)$.

Now, both $\langle\,,\,\rangle$ and $\langle\,,\,\rangle_P$ are ρ'–invariant. By Lemma 6.4, it follows that $\langle\,,\,\rangle_P = t\langle\,,\,\rangle$ for some $t \in \mathbb{R}^+$. The matrix $U = \frac{1}{\sqrt{t}}P$ still satisfies $U^{-1}\rho U = \rho'$, and $U \in U(n)$ because

$$\langle Uv, Uw\rangle = \langle \frac{1}{\sqrt{t}}Pv, \frac{1}{\sqrt{t}}Pw\rangle = \frac{1}{t}\langle Pv, Pw\rangle$$
$$= \frac{1}{t}\langle v, w\rangle_P = \frac{1}{t}\left(t\langle v, w\rangle\right) = \langle v, w\rangle$$

for any $v, w \in \mathbb{C}^n$. $\qquad\square$

Corollary 6.6. *If $\rho, \rho'\colon G \to U(n)$ are irreducible projective unitary representations of a discrete group G, then $\rho \sim_{\mathrm{GL}} \rho' \iff \rho \sim_U \rho'$.*

Proof. Again, the direction $\rho \sim_U \rho' \implies \rho \sim_{\mathrm{GL}} \rho'$ is immediate, so we assume that $\rho \sim_{\mathrm{GL}} \rho'$. This means that for some matrix $P \in \mathrm{GL}_n(\mathbb{C})$ and some function $\lambda\colon G \to \mathbb{C}^*$, we have

$$(4) \qquad\qquad\qquad P\rho(g)P^{-1} = \lambda(g)\rho'(g)$$

for all $g \in G$.

We claim that $\lambda(g) \in S^1$ for each $g \in G$. If g has order m, then raising both sides of (4) to the mth power shows that $P\rho(g)^m P^{-1} = \lambda(g)^m \rho'(g)^m$. Since ρ and ρ' are projective unitary representations, we have $\rho(g)^m, \rho'(g)^m \in S^1$, so $P\rho(g)^m P^{-1} = \rho(g)^m$ and hence $\lambda(g)^m = \rho(g)^m/\rho'(g)^m \in S^1$. This implies that $\lambda(g)$ itself lies in S^1, as desired.

Now, setting $\rho''(g) = \lambda(g)\rho'(g)$, we see that $\rho''(g)\colon G \to U(n)$ is still a projective *unitary* representation, and now $\rho \approx_{\mathrm{GL}} \rho''$ (via the matrix $P \in \mathrm{GL}_n(\mathbb{C})$). By Proposition 6.5, there exists a matrix $U \in U(n)$ with $U\rho(g)U^{-1} = \rho''(g) = \lambda(g)\rho'(g)$ for all $g \in G$, so $\rho(g) \sim_U \rho'(g)$, as desired. $\square$

Using Corollary 6.6, we will deduce a key finiteness result for projective unitary representations of *finite* groups. This will be a corollary of the following classical result due to Schur [17]. Discussions of this result may be found in Karpilovsky [7] and Tappe [20, Corollary 3.6].

Theorem 6.7 (Schur). *For any finite group G, the number of projective equivalence classes ($\sim_{\mathrm{GL}}$–classes) of irreducible projective representations $G \to \mathrm{GL}_n(\mathbb{C})$ is finite.*

We sketch the argument, originally due to Schur [17]. Each projective representation $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ corresponds to a cohomology class in $H^2(G; \mathbb{C}^*)$. Specifically, given $g, h \in G$, we have $\sigma(g,h) := \rho(g)\rho(h)\rho(gh)^{-1} \in \mathbb{C}^*$, and the assignment $(g,h) \mapsto \sigma(g,h)$ is a $\mathbb{C}^*$–valued 2–cocycle on G. If $\rho \sim_{GL} \rho'$, then the corresponding cocyles are cohomologous [7, Chapter 3, Lemma 1.1 (i)], so we have a well-defined class in $H^2(G; \mathbb{C}^*)$ associated to each $\sim_{\mathrm{GL}}$–class of projective representations. Now, for any finite group G, it turns out that the group $H^2(G; \mathbb{C}^*)$ is finite (for a short proof, see [7, Chapter 2, Theorem 3.2]).

Schur showed that each cohomology class contains only finitely many $\sim_{\mathrm{GL}}$–classes of *irreducible* projective representations.[1] This is proven in three steps. First, one observes that $\approx_{\mathrm{GL}}$–classes of projective representations with cocycle σ are in bijection with isomorphism classes of modules over the *twisted group algebra* $\mathbb{C}^\sigma[G]$, and irreducible projective representations correspond to irreducible modules. Here $\mathbb{C}^\sigma[G]$ is the $\mathbb{C}$–algebra with basis $\{\overline{g} \,|\, g \in G\}$ and with multiplication induced by setting $\overline{g_1 g_2} = \sigma(g_1, g_2)(\overline{g_1} \cdot \overline{g_2})$ (for more detail on twisted group algebras, see [7, Section 3.2]).

The second step is to show that for any cocycle σ, there are finitely many isomorphism classes of irreducible modules over $\mathbb{C}^\sigma[G]$. In fact, these isomorphism classes are in bijection with the so-called σ–regular conjugacy classes in G [7, Theorem 6.7]. (An extension of this result can be found in Tappe [20].) We now conclude that there are finitely many $\approx_{\mathrm{GL}}$–classes of irreducible projective representations with cocycle σ. The final step is to check that if σ' is cohomologous to σ, then every projective representation with cocyle σ' is $\sim_{\mathrm{GL}}$–equivalent to a projective representation with cocycle σ ([7, Lemma 1.1 (ii)]). This shows that there are finitely many $\sim_{\mathrm{GL}}$–classes of irreducible projective representations with associated cohomology class $[\sigma] \in H^2(G; \mathbb{C}^*)$.

Remark 6.8. *For ordinary representations of finite groups, the fact that there are finitely many irreducibles is often proven by observing that the group ring $\mathbb{C}[G]$ is semisimple. If G is solvable, then a theorem of Passman [11, Theorem 3] asserts that $\mathbb{C}^\sigma[G]$ is semisimple. For our main results, we will only consider the case where G is* abelian*, so Passman's result is sufficient for our purposes. In general, one could try to prove this result by the same averaging argument used to show that the ordinary group ring $\mathbb{C}[G]$ is semisimple (see, for example, Serre [19, Chapter 6,*

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[1]In fact, when G is abelian there is at most one $\sim_{\mathrm{GL}}$–class per cohomology class, as proven by Frucht [4] and, for more general fields, by Yamazaki [22, Section 6.1].

Proposition 9]*). However, in order to average over the group G, one must divide by $\sum_{g \in G} \alpha(g, g^{-1}) \in \mathbb{C}^*$ rather than by the order of G. It is unclear when this sum is non-zero.*

Combining Corollary 6.6 and Theorem 6.7 yields a finiteness result for irreducible projective unitary representations of finite groups.

Corollary 6.9. *For any finite group G, there are finitely many projective* unitary *equivalence classes ($\sim_U$–classes) of irreducible projective representations $\rho\colon G \to U(n)$. Equivalently, there are finitely many irreducible elements in the moduli space $\mathrm{Hom}(Q, \mathrm{PU}(n))/\mathrm{PU}(n)$.*

Corollary 6.10. *The space $\mathrm{Hom}_A(\Gamma, U(n))$ is the disjoint union (topologically) of the subspaces $\mathrm{Hom}_A(\Gamma, U(n)) \cap \mathrm{Sum}(\Gamma, U(n))$ and*

$$\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]} \colon = \{\rho\colon \Gamma \to U(n) \mid \rho(A) \subset S^1,\ \overline{\rho} \cong \psi\colon Q \to \mathrm{PU}(n)\},$$

where ψ ranges over the finite set of irreducibles in $\mathrm{Hom}(Q, \mathrm{PU}(n))/\mathrm{PU}(n)$.

Proof. By Lemma 6.3, $\mathrm{Hom}_A(\Gamma, U(n))$ is the disjoint union, set-theoretically, of the above spaces. Lemma 3.2 tells us that $\mathrm{Sum}(\Gamma, U(n))$ is closed in $\mathrm{Hom}(\Gamma, U(n))$, so $\mathrm{Hom}_A(\Gamma, U(n)) \cap \mathrm{Sum}(\Gamma, U(n))$ is closed in $\mathrm{Hom}_A(\Gamma, U(n))$. Each subspace $\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}$ is closed in $\mathrm{Hom}_A(\Gamma, U(n))$ as well, because it is the inverse image of the point $[\psi]$ under the continuous map $\mathrm{Hom}_A(\Gamma, U(n)) \to \mathrm{Hom}(Q, U(n))/U(n)$ sending ρ to $[\overline{\rho}]$.

We have now partitioned $\mathrm{Hom}_A(\Gamma, U(n))$ into a finite number of disjoint closed sets, and hence each must be open as well, completing the proof. $\square$

We now study the subspaces $\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}$. Each of these subspaces is the union, over $\psi' \cong \psi$, of the subspaces

$$\mathrm{Hom}_A(\Gamma, U(n))_{\psi'} := \{\rho \in \mathrm{Hom}_A(\Gamma, U(n)) \mid \overline{\rho} = \psi'\}.$$

If $\psi' = P\psi P^{-1}$ for some $P \in \mathrm{PU}(n)$, then P and P^{-1} induce inverse homeomorphisms between $\mathrm{Hom}_A(\Gamma, U(n))_{\psi}$ and $\mathrm{Hom}_A(\Gamma, U(n))_{\psi'}$. In fact, as we now show, $\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}$ is a locally trivial fiber bundle with these fibers.

Proposition 6.11. *Let $\psi\colon Q \to \mathrm{PU}(n)$ be a homomorphism for which the subspace $\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}$ is non-empty. Then the map*

$$\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]} \xrightarrow{\ \pi\ } \mathrm{PU}(n) \cdot \psi \subset \mathrm{Hom}(Q, \mathrm{PU}(n)),$$

given by $\pi(\rho) = \overline{\rho}$, is a fiber bundle over the orbit of ψ, with fiber $\mathrm{Hom}_A(\Gamma, U(n))_{\psi}$.

Proof. Let $\overline{\mathrm{Stab}}(\psi) \subset \mathrm{PU}(n)$ be the stabilizer of ψ under the $\mathrm{PU}(n)$–action, and let $\mathrm{Stab}(\psi)$ denote the inverse image of $\overline{\mathrm{Stab}}(\psi)$ in $U(n)$. Note that both of these subgroups are closed. Since $U(n)$ is a compact Lie group and $\mathrm{Stab}(\psi)$ is a closed subgroup, the quotient map $q_\psi\colon U(n) \to U(n)/\mathrm{Stab}(\psi)$ is a principal $\mathrm{Stab}(\psi)$–bundle (see, for example, Duistermaat and Kolk [3, 1.10.7 and 1.11.4]). Hence q_ψ admits local sections. Choose an open covering $\{V_i\}_i$ of $U(n)/\mathrm{Stab}(\psi)$ admitting local sections $\alpha_i\colon V_i \to U(n)$. Since $U(n)/\mathrm{Stab}(\psi)$ is a Lie group, it is regular (in fact, quotients of regular spaces by compact groups are always regular), so we may choose an open cover $\{U_j\}_j$ of $U(n)/\mathrm{Stab}(\psi)$ such that for each j there exists i with $\overline{U_j} \subset V_i$.

We have homeomorphisms

$$U(n)/\text{Stab}(\psi) \overset{\cong}{\to} \text{PU}(n)/\overline{\text{Stab}}(\psi) \cong \text{PU}(n) \cdot \psi,$$

which we treat as identifications. So we will consider $\{U_j\}_j$ and $\{V_i\}_i$ as coverings of $\text{PU}(n)/\overline{\text{Stab}}(\psi)$ and $\text{PU}(n) \cdot \psi$.

We claim that π is trivial over each open set U_j. In fact, we will show that π is trivial over the closures of the U_j. Since $\overline{U_j} \subset V_i$ for some $i = i(j)$, if we set $\beta_j = \alpha_i$ then β_j is a section of q_ψ over $\overline{U_j}$. We have fiber-preserving maps

$$\phi_j \colon \overline{U_j} \times \text{Hom}_A(\Gamma, U(n))_\psi \longrightarrow \pi^{-1}(\overline{U_j})$$

given by

$$\phi_j(u, \rho) = \beta_j(u)\rho\beta_j(u)^{-1},$$

and we claim these maps are homeomorphisms. Note that continuity of ϕ_j follows immediately from the fact that β_j and the action of $U(n)$ on $\text{Hom}(\Gamma, U(n))$ are continuous.

The space $\overline{U_j}$ is compact, since it is closed in the compact space $\text{PU}(n)/\overline{\text{Stab}}(\psi)$, and $\text{Hom}_A(\Gamma, U(n))_\psi$ is compact because it is closed in $\text{Hom}_A(\Gamma, U(n))$, which is compact by Lemma 6.2. So the domain of ϕ_j is compact, and it will suffice to check that ϕ_j is a bijection.

First we check that ϕ_j is surjective. Consider a representation $\rho \in \pi^{-1}(\overline{U_j})$. Then $\overline{\rho} = [x]\psi[x]^{-1}$ for some $x \in U(n)$ with $q_\psi(x) \in \overline{U_j}$ (where $\overline{U_j}$ is viewed as a subspace of $U(n)/\text{Stab}(\psi)$). Let $u = q_\psi(x) \in \overline{U_j}$. Now $\beta_j(u) = xk$ for some $k \in \text{Stab}(\psi)$, and we have

$$[\beta_j(u)]\psi[\beta_j(u)]^{-1} = [x][k]\psi[k]^{-1}[x]^{-1} = [x]\psi[x]^{-1} = \overline{\rho},$$

or in other words $\psi = [\beta_j(u)]^{-1}(\overline{\rho})[\beta_j(u)]$. Now $\beta_j(u)^{-1}\rho\beta_j(u) \in \text{Hom}_A(\Gamma, U(n))_\psi$, and $\phi_j(u, \beta_j(u)^{-1}\rho\beta_j(u)) = \rho$.

Next, we check that ϕ_j is injective. If $\phi_j(u, \rho) = \phi_j(u', \rho')$, then

$$(5) \qquad \beta_j(u')^{-1}\beta_j(u)\rho\beta_j(u)^{-1}\beta_j(u') = \rho'.$$

Since $\overline{\rho} = \overline{\rho'} = \psi$, we see that $\beta_j(u')^{-1}\beta_j(u) \in \text{Stab}(\psi)$. But since β_j is a section of q_ψ, this implies that $u' = u$, and by (5), we have $\rho' = \rho$ as well, proving injectivity. $\qquad\square$

We will now analyze the individual fibers $\text{Hom}_A(\Gamma, U(n))_\psi$ of the bundles from Proposition 6.11. Each such subspace admits a restriction map to $\text{Hom}(A, S^1)$. Note that when A is a finitely generated abelian group, $\text{Hom}(A, S^1)$ is a disjoint union of tori of dimension $\text{rk}(A)$.

Proposition 6.12. *For each $\psi \colon Q \to U(n)$, the restriction map*

$$R \colon \text{Hom}_A(\Gamma, U(n))_\psi \longrightarrow \text{Hom}(A, S^1)$$

has the structure of a (non-surjective) finite covering map with structure group $\text{Hom}(Q, S^1)$.

Proof. The action of $\text{Hom}(Q, S^1)$ on $\text{Hom}_A(\Gamma, U(n))_\psi$ is given by

$$(\chi \cdot \rho)(\gamma) = \chi([\gamma])\rho(\gamma).$$

Since $\chi([\gamma])$ is central in $U(n)$, $\chi \cdot \rho$ is a homomorphism with $(\chi \cdot \rho)(A) \subset S^1$, so we have a well-defined action. Moreover, $\overline{\chi \cdot \rho} = \overline{\rho} = \psi$ and the action is free because

$\chi \cdot \rho(\gamma) = \rho(\gamma)$ implies that $\chi([\gamma]) = 1$ for all $\gamma \in \Gamma$. Hence the quotient map for this action is a covering map whose structure group is the finite group $\mathrm{Hom}(Q, S^1)$.

If $a \in A$, then $(\chi \cdot \rho)(a) = \chi(1)\rho(a) = \rho(a)$, so the restriction map R factors through the quotient space for this action. We must show that the induced map

$$\Phi \colon (\mathrm{Hom}_A(\Gamma, U(n))_\psi) / \mathrm{Hom}(Q, S^1) \longrightarrow R(\mathrm{Hom}_A(\Gamma, U(n))_\psi)$$

is a homeomorphism. We claim that $\mathrm{Hom}_A(\Gamma, U(n))_\psi$ is compact. This space is the inverse image of $\psi \in \mathrm{Hom}(Q, \mathrm{PU}(n))$ under the map

$$\mathrm{Hom}_A(\Gamma, U(n)) \longrightarrow \mathrm{Hom}(Q, \mathrm{PU}(n)),$$

so $\mathrm{Hom}_A(\Gamma, U(n))_\psi$ is closed in $\mathrm{Hom}_A(\Gamma, U(n))$, and $\mathrm{Hom}_A(\Gamma, U(n))$ is compact by Lemma 6.2. It now follows that the domain of Φ is compact, so to show that Φ is a homeomorphism we need only show that it is injective (Φ is surjective by definition).

Say $\Phi(\rho) = \Phi(\rho')$. Then we know that $\overline{\rho} = \overline{\rho'} = \psi$ and $\rho|_A = \rho'|_A$. The first condition implies that for any $\gamma \in \Gamma$, we have $\rho(\gamma) = \lambda(\gamma)\rho'(\gamma)$, for some $\lambda(\gamma) \in S^1$, and the second condition implies that $\lambda(a) = 1$ if $a \in A$. We simply need to check that $\lambda \colon \Gamma \to S^1$ is a homomorphism. For any $\gamma \in \Gamma$, we have $\lambda(\gamma) = \rho(\gamma)\rho'(\gamma)^{-1}$. Now

$$\begin{aligned}
\lambda(\gamma_1\gamma_2) &= \rho(\gamma_1\gamma_2)\rho'(\gamma_1\gamma_2)^{-1} = \rho(\gamma_1)\rho(\gamma_2)\rho'(\gamma_2)^{-1}\rho'(\gamma_1)^{-1} \\
&= \rho(\gamma_1)\lambda(\gamma_2)\rho'(\gamma_1)^{-1} = \rho(\gamma_1)\rho'(\gamma_1)^{-1}\lambda(\gamma_2) \\
&= \lambda(\gamma_1)\lambda(\gamma_2).
\end{aligned}$$

$\square$

Next, we need the following fact regarding the images of the restriction maps R.

Proposition 6.13. *If A is a finitely generated abelian group and Q is abelian, then for each homomorphism $\psi \colon Q \to \mathrm{PU}(n)$, the image of $\mathrm{Hom}_A(\Gamma, U(n))_\psi$ in $\mathrm{Hom}(A, S^1)$ is a closed subvariety.*

The proof depends on the existence of a particular type of presentation for the groups in question.

Lemma 6.14. *Consider an extension of groups*

$$1 \to A \to \Gamma \to Q \to 1$$

in which A is a finitely generated abelian group and Q is finite. If Q has a presentation of the form

$$Q = \langle q_1 \ldots, q_l \mid r_1(q_1, \ldots, q_l) = \ldots = r_p(q_1, \ldots, q_l) = 1 \rangle,$$

for some words r_i in the free group on l letters, and A has a presentation of the form

$$\langle a_1 \ldots, a_{k+m} \mid a_i a_j = a_j a_i \, \forall i, j \quad and \quad a_1^{n_1} = \cdots = a_m^{n_m} = 1 \rangle,$$

then for some words w_i and u_{ij} in the free group on $k + m$ letters, then

(6)
$$\begin{aligned}
\langle \alpha_1, \ldots, \alpha_{k+m}, \gamma_1 \ldots, \gamma_l \mid \ & r_i(\gamma_1 \ldots, \gamma_l) = w_i(\alpha_1, \ldots, \alpha_{k+m}) \, \forall i, \\
& \alpha_i \alpha_j = \alpha_j \alpha_i \, \forall i, j, \ \alpha_j^{n_j} = 1, \, j = 1, \ldots, m, \\
& \gamma_i \alpha_j \gamma_i^{-1} = u_{ij}(\alpha_1, \ldots, \alpha_{k+m}) \, \forall i, j \, \rangle,
\end{aligned}$$

is a presentation for Γ.

Note that the words u_{ij} and w_i are not unique.

Proof. To begin, we must specify the words u_{ij} and w_i. Choose elements $\tilde{q}_i \in \Gamma$ lying over $q_i \in Q$. Since A is normal in Γ, we know that $\tilde{q}_i a_j \tilde{q}_i^{-1} \in A$, and hence $\tilde{q}_i a_j \tilde{q}_i^{-1} = u_{ij}(a_1, \ldots, a_{k+m})$ for some word u_{ij}. Next, since $r_i(q_1, \ldots, q_l) = 1$ in Q, we know that $r_i(\tilde{q}_1, \ldots, \tilde{q}_l) \in A$, and hence $r_i(\tilde{q}_1, \ldots, \tilde{q}_l) = w_i(a_1, \ldots, a_)$ for some word w_i.

Now, let $\widetilde{\Gamma}$ denote the group presented by (6), and let $\widetilde{A}$ denote the subgroup generated by $\alpha_1, \ldots, \alpha_{k+m}$. Let $\Phi \colon \widetilde{\Gamma} \to \Gamma$ be the homomorphism defined by $\Phi(\alpha_i) = a_i$ and $\Phi(\gamma_i) = \tilde{q}_i$. Then Φ is surjective, and its restriction to $\widetilde{A}$ is a surjection onto $A < \Gamma$. The third set of relations in (6) ensure that $\widetilde{A}$ is normal in $\widetilde{\Gamma}$, and we define $\widetilde{Q} = \widetilde{\Gamma}/\widetilde{A}$. The map Φ induces a surjection $\overline{\Phi} \colon \widetilde{Q} \twoheadrightarrow Q$, and we have a commutative diagram

(7)
$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{A} & \longrightarrow & \widetilde{\Gamma} & \longrightarrow & \widetilde{Q} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \overline{\Phi}} & & \\
1 & \longrightarrow & A & \longrightarrow & \Gamma & \longrightarrow & Q & \longrightarrow & 1.
\end{array}
$$

The map $\widetilde{\Gamma} \to \widetilde{Q}$ induces a surjection from the free group on the generators γ_i onto $\widetilde{Q}$, and this surjection factors through the quotient group

$$\langle \gamma_1, \ldots, \gamma_l \mid r_i(\gamma_1, \ldots, \gamma_l) = 1 \, \forall i \rangle \cong Q.$$

Hence we have a surjection $Q \twoheadrightarrow \widetilde{Q}$, meaning that $\widetilde{Q}$ is a finite group of order at most $|Q|$. The existence of the surjection $\widetilde{\Phi} \colon \widetilde{Q} \twoheadrightarrow Q$ now shows that both of these surjections must in fact be isomorphisms.

Next, we show that the map $\widetilde{A} \to A$ is injective. Each element $\alpha \in \widetilde{A}$ has the form $\alpha_1^{p_1} \alpha_2^{p_2} \cdots \alpha_{k+m}^{p_{k+m}}$ for some $p_i \in \mathbb{Z}$. Our presentation for A shows that if $\Phi(\alpha) = 0$, then p_i is a multiple of n_i for $1 \leqslant i \leqslant m$, and $p_i = 0$ for $i > m$. But such elements are already trivial in $\widetilde{\Gamma}$, so Φ is injective when restricted to $\widetilde{A}$.

We have now shown that the two outer maps in (7) are isomorphisms, and the 5-lemma shows that Φ is an isomorphism as well. $\qquad\square$

Proof of Proposition 6.13. The image of $\mathrm{Hom}_A(\Gamma, U(n))_\psi$ in $\mathrm{Hom}(A, S^1)$ is the set of representations $\rho \colon A \to S^1$ which can be extended to representations $\widetilde{\rho} \colon \Gamma \to U(n)$ satisfying $[\widetilde{\rho}(\gamma)] = \psi[\gamma] \in \mathrm{PU}(n)$ for all $\gamma \in \Gamma$.

Since Q is a finite abelian group, it has a presentation of the form

$$Q = \langle q_1, \ldots, q_l \mid q_i q_j = q_j q_i \, \forall i, j, \ q_i^{r_i} = 1 \, \forall i \rangle$$

for some $r_i \in \mathbb{N}$. Hence Lemma 6.14 yields a presentation for Γ of the form

(8)
$$
\begin{aligned}
\langle \alpha_1, \ldots, \alpha_{k+m}, \gamma_1 \ldots, \gamma_l | \ & \gamma_i \gamma_j \gamma_i^{-1} \gamma_j^{-1} = w_{ij}(\alpha_1, \ldots, \alpha_{k+m}) \, \forall i, j \\
& \gamma_i^{r_i} = w_i(\alpha_1, \ldots, \alpha_{k+m}) \forall i, \\
& \alpha_i \alpha_j = \alpha_j \alpha_i \, \forall i, j, \ a_j^{n_j} = 1, \ j = 1, \ldots, m, \\
& \gamma_i \alpha_j \gamma_i^{-1} = u_{ij}(\alpha_1, \ldots, \alpha_{k+m}) \, \forall i, j \rangle,
\end{aligned}
$$

for some words w_i, w_{ij}, and u_{ij} in the free group on $k + m$ letters, and some $n_i \in \mathbb{N}$.

Now, say $\rho \colon A \to S^1$ lies in $\mathrm{Hom}(A, S^1)$. Extending ρ to an representation $\widetilde{\rho} \in \mathrm{Hom}_A(\Gamma, U(n))_\psi$ amounts to choosing elements $\widetilde{\rho}(\gamma_i) \in U(n)$ such that

(i) $[\widetilde{\rho}(\gamma_i)] = \psi([\gamma_i])$
(ii) $\widetilde{\rho}(\gamma_i)\widetilde{\rho}(\gamma_j)\widetilde{\rho}(\gamma_i)^{-1}\widetilde{\rho}(\gamma_j)^{-1} = w_{ij}(\rho(\alpha_1), \ldots, \rho(\alpha_k)) \in S^1$
(iii) $\widetilde{\rho}(\gamma_i)^{r_i} = w_i(\rho(\alpha_1), \ldots, \rho(\alpha_k)) \in S^1.$
(iv) $\widetilde{\rho}(\gamma_i)\rho(\alpha_j)\widetilde{\rho}(\gamma_i)^{-1} = u_{ij}(\rho(\alpha_1), \ldots, \rho(\alpha_k)) \in S^1.$

Choose $\psi_i \in U(n)$ such that $[\psi_i] = \psi([\gamma_i]) \in \mathrm{PU}(n)$. Since $[\gamma_i][\gamma_j][\gamma_i^{-1}][\gamma_j^{-1}] = 1$ in Q, we must have $\psi_i\psi_j\psi_i^{-1}\psi_j^{-1} \in S^1$. We set $c_{ij} = \psi_i\psi_j\psi_i^{-1}\psi_j^{-1}$. Note that since the ψ_i are well-defined up to multiplication by scalars in S^1, the c_{ij} depend only on $\psi\colon Q \to \mathrm{PU}(n)$.

If $\rho\colon A \to S^1$ extends to $\widetilde{\rho} \in \mathrm{Hom}_A(\Gamma, U(n))_\psi$, then Equation (i) implies that $\widetilde{\rho}(\gamma_i) = \lambda_i\psi_i$ for some $\lambda_i \in S^1$ ($i = 1, \ldots, l$), so by Equation (ii), we must have

$$(9) \qquad\qquad w_{ij}(\rho(\alpha_1), \ldots, \rho(\alpha_k)) = c_{ij} \in S^1.$$

Since $\rho(\alpha_j) \in S^1$, Equation (iv) implies that

$$(10) \qquad\qquad \rho(\alpha_j) = u_{ij}(\rho(\alpha_1), \ldots, \rho(\alpha_k)).$$

Hence the image of $\mathrm{Hom}_A(\Gamma, U(n))_\psi$ in $\mathrm{Hom}(A, S^1)$ is contained in the subvariety

$$(11) \quad \begin{aligned}\mathrm{Hom}(A, S^1)_\psi^\Gamma := \{\rho\colon\ &A \to S^1 \,|\, \rho(\alpha_j) = u_{ij}(\rho(\alpha_1), \ldots, \rho(\alpha_k)) \,\forall i, j,\\ &\text{and } w_{ij}(\rho(\alpha_1), \ldots, \rho(\alpha_k)) = c_{ij}\}.\end{aligned}$$

cut out by (9) and (10). We claim that the image is in fact *equal* to $\mathrm{Hom}(A, S^1)_\psi^\Gamma$.

Say $\rho \in \mathrm{Hom}(A, S^1)_\psi^\Gamma$. We must show that for some $\xi_i \in S^1$, the assignments $\widetilde{\rho}(\gamma_i) = \xi_i\psi_i$, $\widetilde{\rho}(\alpha_i) = \rho(\alpha_i)$ produce a homomorphism $\widetilde{\rho}\colon \Gamma \to U(n)$. The definition of $\mathrm{Hom}(A, U(n))_\psi^\Gamma$ guarantees that for *any* choice of ξ_i, these elements satisfy Equations (i), (ii), and (iv) above. Since $[\gamma_i]^{r_i} = 1$ in Q, we know that $\psi_i^{r_i} \in S^1$. If we choose ξ_i to be an n_i^{th} root of $w_i(\rho(\alpha_1), \ldots, \rho(\alpha_k)) \cdot \psi_i^{-r_i} \in S^1$, then the $\widetilde{\rho(\gamma_i)}$ will also satisfy (iii). $\qquad\square$

Remark 6.15. *We have shown that the image of* $\mathrm{Hom}_A(\Gamma, U(n))_\psi$ *in* $\mathrm{Hom}(A, S^1)$ *is precisely the subvariety* $\mathrm{Hom}(A, S^1)_\psi^\Gamma$ *defined by (11). The equations defining this subvariety depend only on our presentation for* Γ *and the numbers* $c_{ij} = c_{ij}(\psi)$. *One may check that* $c_{ij}(\psi) = c_{ij}(X\psi X^{-1})$ *for any* $X \in \mathrm{PU}(n)$, *so the subvariety* $\mathrm{Hom}(A, S^1)_\psi^\Gamma$ *depends only on the isomorphism class of* ψ.

7. The dimension of the moduli space of irreducible representations

Throughout this section, Γ will denote an infinite discrete group sitting in an extension

$$1 \longrightarrow A \longrightarrow \Gamma \longrightarrow Q$$

with A a finitely generated abelian group of rank $k > 0$ and Q a finite abelian group.

7.1. Triangulations.

Lemma 7.1. *The moduli space* $\mathrm{Hom}(\Gamma, U(n))/U(n)$ *admits a triangulation in which* $\mathrm{Sum}(\Gamma, U(n))/U(n)$ *and each subset* $\left(\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}\right)/U(n)$, *with* ψ *irreducible, are subcomplexes.*

Modding out $\mathrm{Sum}(\Gamma, U(n))/U(n)$ *then yields a CW–structure on* $\mathrm{Irr}_n^+(\Gamma)$ *in which each* $\left(\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}\right)/U(n)$ *is a subcomplex, and* $+$ *is a vertex.*

Proof. By Illman's theorem on equivariant triangulations ([6, Theorem B]) there exists a $U(n)$–equivariant triangulation of $\mathrm{Hom}(\Gamma, U(n))$ in which the subvariety $\mathrm{Hom}_A(\Gamma, U(n))$ is a subcomplex. By Schur's Lemma, $\mathrm{Sum}(\Gamma, U(n))$ is precisely the set of representations whose stabilizer (in $U(n)$) is *larger* than S^1. Hence $\mathrm{Sum}(\Gamma, U(n))$ must be a subcomplex as well. By Corollary 6.10, modding out $U(n)$ gives the desired triangulation of $\mathrm{Hom}(\Gamma, U(n))/U(n)$.

Collapsing $\mathrm{Sum}(\Gamma, U(n))/U(n) \subset \mathrm{Hom}(\Gamma, U(n))/U(n)$ to a point now provides the desired cell structure on $\mathrm{Irr}_n^+(\Gamma)$. $\qquad\square$

We will need a simple consequence of Brower's Invariance of Domain theorem.

Lemma 7.2. *Let X be a topological space with open subsets $U, V \subset X$. If $U \cap V \neq \emptyset$ and there exist homeomorphisms $f \colon U \xrightarrow{\cong} \mathbb{R}^n$, $g \colon V \xrightarrow{\cong} \mathbb{R}^m$, then $n = m$.*

Proof. Say $n \geqslant m$. Then $U \cap V$ is open in $U \cong \mathbb{R}^n$, and hence contains a subset $U' \cong \mathbb{R}^n$ with U' open in X. If $i \colon \mathbb{R}^m \to U' \cong \mathbb{R}^n$ denotes the usual coordinate-inclusion, we have an injective, continuous map

$$\mathbb{R}^m \xhookrightarrow{i} U' \xhookrightarrow{j} V \xrightarrow{g} \mathbb{R}^m,$$

where j denotes the inclusion. By Invariance of Domain, the image of the composite $g \circ j \circ i$ must be open. Since g is a homeomorphism, $j \circ i(\mathbb{R}^m)$ is open in V, and since U' is open in V, $i(\mathbb{R}^m)$ must be open in U'. But i was the standard coordinate-inclusion of $\mathbb{R}^m$ into $\mathbb{R}^n \cong U'$, so $i(\mathbb{R}^m)$ is a closed subset of U'. Since $U' \cong \mathbb{R}^n$ is connected, no proper subset can be both open and closed, so i must be surjective and we must have $n = m$. $\qquad\square$

Lemma 7.3. *If X is a k–dimensional simplicial complex, then no open set in X is homeomorphic to $\mathbb{R}^m$ with $m > k$. Consequently, a simplicial complex of dimension k cannot be homeomorphic to a simplicial complex of dimension l unless $l = k$.*

Proof. Let D denote the union of the interiors of the maximal simplices in X. Then D is dense in X, and each point in D has an open neighborhood homeomorphic to $\mathbb{R}^l$ with $l \leqslant k$. If $U \subset X$ is open and $U \cong \mathbb{R}^m$, then U contains some point $d \in D$. Let V_d be an open neighborhood of d homeomorphic to $\mathbb{R}^l$ with $l \leqslant k$. Then $U \cap V_d \neq \emptyset$, and Lemma 7.2 implies that $m = l \leqslant k$. $\qquad\square$

Corollary 7.4. *If $\psi \colon Q \to \mathrm{PU}(n)$ is irreducible and $\mathrm{Hom}(Q, \mathrm{PU}(n))_\psi^\Gamma$ is non-empty, then the subspace $\mathrm{Hom}_A(\Gamma, U(n))_\psi$ is homeomorphic to a finite simplicial complex of dimension at most k.*

Proof. By Proposition 6.12, $\mathrm{Hom}_A(\Gamma, U(n))_\psi$ is a finite cover of the subvariety

$$\mathrm{Hom}(A, S^1)_\psi^\Gamma \subset \mathrm{Hom}(A, S^1) \cong (S^1)^k.$$

By basic results on triangulations of algebraic varieties (e.g. Hironaka [5]), there exists a triangulation of $(S^1)^k$ with $\mathrm{Hom}(A, S^1)_\psi^\Gamma$ as a subcomplex. By Lemma 7.3, this triangulation is at most k–dimensional. Since any cover of a simplicial complex has a triangulation of the same dimension, the result follows. Note that all of these triangulations have finitely many cells, since $(S^1)^k$ is compact and the covering is finite. $\qquad\square$

7.2. **The moduli space.**

Theorem 7.5. *For any $n > 0$, there is a triangulation of $\mathrm{Hom}(\Gamma, U(n))/U(n)$ in which $\mathrm{Sum}(\Gamma, U(n))/U(n)$ is a subcomplex and all simplices outside of this subcomplex have dimension at most $k = \mathrm{rk}(A)$. Consequently, the one-point compactification $\mathrm{Irr}_n^+(\Gamma)$ of the moduli space of irreducible $U(n)$–representations of Γ is homeomorphic to a CW–complex of dimension at most $k = \mathrm{rk}(A)$.*

Proof. The proof is by induction on $|Q|$. When $|Q| = 1$, $\Gamma \cong \mathbb{Z}^k$, and hence $\mathrm{Irr}_n^+(\Gamma)$ is a point for $n > 1$. Now, say A has a presentation

$$\langle a_1 \ldots, a_{k+m} \mid a_i a_j = a_j a_i \, \forall i, j \text{ and } a_1^{n_1} = \cdots a_m^{n_m} = 1 \rangle.$$

When $n = 1$, all representations are irreducible, and a homomorphism $\rho \colon A \to S^1$ is determined by the elements $\rho(a_i)$, $i = 1, \ldots, k + m$. For $i = 1, \ldots, m$, we know that $\rho(a_i)$ lies in the discrete set of n_i^{th} roots of unity, while the elements $\rho(a_{m+1}), \ldots, \rho(a_{k+m})$ can be arbitrary elements of S^1. Hence $\mathrm{Hom}(A, S^1) = \mathrm{Hom}(A, S^1)/S^1$ is homeomorphic to a disjoint union of rank k tori $(S^1)^k$, and $\mathrm{Irr}_1^+(A) = \mathrm{Hom}(A, S^1) \coprod \{+\}$. This completes the base case.

We now assume the result for all extensions

$$A' \longrightarrow \Gamma' \longrightarrow Q'$$

(with A finitely generated abelian and Q' finite abelian) such that $|Q'| < |Q|$.

Consider the triangulation of $\mathrm{Hom}(\Gamma, U(n))/U(n)$ produced by Lemma 7.1. We claim that any maximal simplices of dimension greater than k must lie either in $\mathrm{Sum}(\Gamma, U(n))/U(n)$ or in $\left(\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]} \right)/U(n)$ for some irreducible ψ. Let σ be a maximal simplex of dimension $m > k$, and assume σ does not lie in any of these subcomplexes. Then

$$(12) \qquad\qquad \overset{\circ}{\sigma} \subset \bigcup_{A \leqslant H < \Gamma} \mathrm{Irr}_n(\Gamma)_H,$$

where $\overset{\circ}{\sigma}$ is the interior of σ and $\mathrm{Irr}_n(\Gamma)_H$ denotes the subspace of representations induced from H (Definition 5.3). Now, let $\mathrm{Irr}_n^+(\Gamma)_H = \mathrm{Irr}_n(\Gamma)_H \cup \{+\}$. Then $\mathrm{Irr}_n^+(\Gamma)_H$ is the image of the composite map

$$\mathrm{Hom}(H, U(n))/U(n) \xrightarrow{\mathrm{Ind}_H^\Gamma} \mathrm{Hom}(\Gamma, U(n))/U(n) \longrightarrow \mathrm{Irr}_n^+(\Gamma),$$

and since $\mathrm{Hom}(H, U(n))/U(n)$ is compact and $\mathrm{Irr}_n^+(\Gamma)$ is Hausdorff, we see that $\mathrm{Irr}_n^+(\Gamma)_H$ is closed in $\mathrm{Irr}_n^+(\Gamma)$. We now have

$$\overset{\circ}{\sigma} = \bigcup_{A \leqslant H < \Gamma} (\mathrm{Irr}_n^+(\Gamma)_H) \cap \overset{\circ}{\sigma}.$$

Since $\mathrm{Irr}_n^+(\Gamma)_H$ is closed in $\mathrm{Irr}_n^+(\Gamma)$, the intersection $\mathrm{Irr}_n^+(\Gamma)_H \cap \overset{\circ}{\sigma}$ is closed in $\overset{\circ}{\sigma} \cong \mathbb{R}^m$. By the Baire Category Theorem, $\mathrm{Irr}_n^+(\Gamma)_{H_0} \cap \overset{\circ}{\sigma}$ must have non-empty interior (as a subset of $\overset{\circ}{\sigma}$) for some subgroup H_0. Thus $\mathrm{Irr}_n^+(\Gamma)_{H_0} \cap \overset{\circ}{\sigma}$ contains a subset V, open in $\overset{\circ}{\sigma}$, such that V is homeomorphic to $\mathbb{R}^m$. Since σ is maximal, $\overset{\circ}{\sigma}$ is open in $\mathrm{Irr}_n^+(\Gamma)$, and hence V is open in $\mathrm{Irr}_n^+(\Gamma)$ as well. Thus V is open as a subset of $\mathrm{Irr}_n^+(\Gamma)_{H_0}$. In fact, V cannot contain the basepoint $+$, since $+$ is a vertex and cannot lie in the interior of a cell of dimension $m > k > 0$. Hence V is an open subset of $\mathrm{Irr}_n(\Gamma)_{H_0}$. By Proposition 5.5, induction induces a covering map

$$\mathrm{Ind}_{H_0}^\Gamma \colon \mathrm{Irr}_{n/r}(H_0)^{\mathrm{free}} \twoheadrightarrow \mathrm{Irr}_{\frac{n}{r}}(\Gamma)_{H_0},$$

(where r is the index of H_0 in Γ) and by shrinking V if necessary, we can find an open subset $V' \subset \mathrm{Irr}_{n/r}(\Gamma)_{H_0}$ whose inverse image in $\mathrm{Irr}_{n/r}(H_0)^{\mathrm{free}}$ is a disjoint union of open sets homeomorphic to $\mathbb{R}^m$. Since $\mathrm{Irr}_{n/r}(H_0)^{\mathrm{free}}$ is open in $\mathrm{Hom}(H_0, U(\frac{n}{r}))/U(\frac{n}{r})$ (Lemma 5.4), these sets are open in $\mathrm{Hom}(H_0, U(\frac{n}{r}))/U(\frac{n}{r}$ as well. Thus there exists an open neighborhood $U \cong \mathbb{R}^m$ of $\mathrm{Hom}(H_0, U(\frac{n}{r}))/U(\frac{n}{r})$ lying outside $\mathrm{Sum}(H_0, U(\frac{n}{r}))/U(\frac{n}{r})$. By our induction hypothesis, there exists a triangulation of $\mathrm{Hom}(H_0, U(\frac{n}{r}))/U(\frac{n}{r})$ with $\mathrm{Sum}(H_0, U(\frac{n}{r}))/U(\frac{n}{r})$ as a subcomplex and such that all simplices outside this subcomplex have dimension at most k. Hence each point $x \in U$ lies in a maximal simplex τ in $\mathrm{Hom}(H_0, U(\frac{n}{r}))/U(\frac{n}{r})$ of dimension at most k, and U must intersect $\overset{\circ}{\tau}$ non-trivially. But $\overset{\circ}{\tau}$ is an open subset of $\mathrm{Hom}(H_0, U(\frac{n}{r}))/U(\frac{n}{r})$ homeomorphic to $\mathbb{R}^l$ for some $l \leqslant k$, and $U \cong \mathbb{R}^m$ with $m > k$. This contradicts Lemma 7.2.

Hence any maximal simplex of $\mathrm{Hom}(\Gamma, U(n))/U(n)$ with dimension greater than k must lie in $\mathrm{Sum}(\Gamma, U(n))/U(n)$ or in $\left(\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}\right)/U(n)$ for some irreducible ψ. Now, if σ is *any* simplex of $\mathrm{Hom}(\Gamma, U(n))/U(n)$ of dimension greater than k, then σ is contained in some maximal simplex τ, and since τ lies in one of these subcomplexes, so must σ. So all simplices in $\mathrm{Hom}(\Gamma, U(n))/U(n)$ of dimension greater than k lie in one of these subcomplexes.

Let $\sigma \subset \left(\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}\right)/U(n)$ (ψ irreducible) be a maximal simplex of this subcomplex. We need to show that the dimension m of σ is at most k. Each point in σ corresponds to an irreducible representation ρ, and Schur's Lemma tells us that the stabilizer of ρ in $U(n)$ is just S^1. Thus the inverse image of σ in $\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]} \subset \mathrm{Hom}(\Gamma, U(n))$ is simply the $U(n)$–simplex $\sigma \times \mathrm{PU}(n)$. Since σ was maximal in $\mathrm{Irr}_n^+(\Gamma)$, its interior $\overset{\circ}{\sigma}$ is open in $\mathrm{Irr}_n^+(\Gamma)$, and the inverse image of $\overset{\circ}{\sigma}$ in $\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}$ is an open set homeomorphic to $\mathbb{R}^m \times \mathrm{PU}(n)$. Since $\mathrm{PU}(n)$ is a manifold of dimension $n^2 - 1$, this yields an open subset of $\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}$ homeomorphic to $\mathbb{R}^{m+n^2-1}$.

We have shown that each set $\mathrm{Hom}_A(\Gamma, U(n))_\psi$ is homeomorphic to a simplicial complex of dimension at most k (Corollary 7.4). Moreover, by Proposition 6.11, $\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}$ is a locally trivial fiber bundle over the manifold $\mathrm{PU}(n)/\mathrm{Stab}(\psi)$ with fiber $\mathrm{Hom}_A(\Gamma, U(n))_\psi$. Since $\mathrm{PU}(n)/\mathrm{Stab}(\psi)$ has dimension at most n^2-1, we can cover $\mathrm{PU}(n)/\mathrm{Stab}(\psi)$ by open subsets homeomorphic to $\mathbb{R}^l$ for some $l \leqslant n^2-1$, over which $\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}$ is trivial. Each fiber $\mathrm{Hom}_A(\Gamma, U(n))_\psi$ contains a dense subset (the interiors of the maximal simplices) in which each point has a neighborhood (open in the fiber) homeomorphic to some $\mathbb{R}^p$ with $p \leqslant k$. Hence we can find dense subset D of $\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}$ such that each point in D has a neighborhood in $\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}$ which is homeomorphic to $\mathbb{R}^j$ for some $j \leqslant k+n^2-1$. Above, we found an open subset of $\mathrm{Hom}_A(\Gamma, U(n))_{[\psi]}$ homeomorphic to $\mathbb{R}^{m+n^2-1}$. By Lemma 7.2, we must have $m+n^2-1 = j \leqslant k+n^2-1$, so $m \leqslant k$. $\square$

8. Periodicity in stable representation theory

We now combine our results on the moduli space $\mathrm{Irr}_n^+(\Gamma)$ with Tyler Lawson's work on deformation K–theory in order to show that $K^{\mathrm{def}}(\Gamma)$ is 2–periodic above dimension $k-2$. We begin by recalling some definitions and results from Lawson [8, 9].

Definition 8.1. *The* deformation representation ring *of* Γ*, denoted* $R^{\mathrm{def}}(\Gamma)$*, is the spectrum* $\mathrm{Sp}(\mathrm{Rep}(\Gamma))$ *associated to the topological abelian monoid*

$$\mathrm{Rep}(\Gamma) := \coprod_{n=0}^{\infty} \mathrm{Hom}(\Gamma, U(n))/U(n).$$

Explicitly, the spaces in the spectrum $\mathrm{Sp}(\mathrm{Rep}(\Gamma))$ are simply

$$\Omega B\mathrm{Rep}(\Gamma), B\mathrm{Rep}(\Gamma), BB\mathrm{Rep}(\Gamma), \dots,$$

where at each stage we form the (simplicial) classifying space $B(-)$ of the previous stage. Note here that if M is an *abelian* topological monoid, so is BM, because the multiplication map $M \times M \to M$ is a *homomorphism* of monoids, and therefore yields a multiplication $BM \times BM \cong B(M \times M) \to BM$.

Theorem 8.2 (Lawson). *There is a homotopy cofiber sequence of spectra*

$$\Sigma^2 K^{\mathrm{def}}(\Gamma) \xrightarrow{\beta} K^{\mathrm{def}}(\Gamma) \longrightarrow R^{\mathrm{def}}(\Gamma),$$

where β *denotes the Bott map in deformation* K*–theory.*

The Bott map β is induced from the ordinary Bott map $\beta\colon \Sigma^2\mathbf{ku} \to \mathbf{ku}$ in connective K–theory by smashing over $\mathbf{ku}$ with $K^{\mathrm{def}}(\Gamma)$. This uses the $\mathbf{ku}$–module structure on $K^{\mathrm{def}}(\Gamma)$ constructed in [8].

Theorem 8.2 shows that periodicity of $K^{\mathrm{def}}(\Gamma)$ is controlled by the homotopy groups $\pi_* R^{\mathrm{def}}(\Gamma)$. These homotopy groups are in turn linked to the homology of the moduli space $\mathrm{Irr}_n^+(\Gamma)$ by the next result.

Theorem 8.3 (Lawson). *There is a tower of fibration sequences of spectra*

$$
\begin{array}{ccccc}
\cdots \longrightarrow & \mathrm{Sp}(\mathrm{Rep}(\Gamma, n-1)) & \longrightarrow & \mathrm{Sp}(Rep(\Gamma, n)) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle q_n} & & \downarrow{\scriptstyle q_n} & \\
& H\mathbb{Z} \wedge \mathrm{Irr}_{n-1}^+(\Gamma) & & H\mathbb{Z} \wedge \mathrm{Irr}_n^+(\Gamma) &
\end{array}
$$

(i.e. the homotopy fiber of each q_i *is* $\mathrm{Sp}(\mathrm{Rep}(\Gamma, i-1))$*) and* $R^{\mathrm{def}}(\Gamma)$ *is the homotopy colimit of the top horizontal sequence.*

In this theorem, $H\mathbb{Z}$ denotes the Eilenberg–MacLane spectrum associated to $\mathbb{Z}$, and $\mathrm{Rep}(\Gamma, n)$ denotes the submonoid of $\mathrm{Rep}(\Gamma)$ generated by representation of dimension at most n. Said another way, $\mathrm{Rep}(\Gamma, n) \subset \mathrm{Rep}(\Gamma)$ consists of representation whose irreducible summands all have dimension at most n. Note that for any CW–complex X, the homotopy groups of the spectrum $H\mathbb{Z} \wedge X$ are simply the (reduced) integral homology groups of X, so

$$\pi_* \left(H\mathbb{Z} \wedge \mathrm{Irr}_n^+(\Gamma) \right) \cong \widetilde{H}_*(\mathrm{Irr}_n^+(\Gamma); \mathbb{Z}).$$

It is useful to note that for a based space X, the spectrum $H\mathbb{Z} \wedge X$ can also be viewed as the spectra associated to the topological abelian monoid $\mathrm{SpSym}^\infty(X)$. As explained above, the zeroth space of this spectrum is $\Omega B\mathrm{Sym}^\infty(X)$, but since $\mathrm{Sym}^\infty(X)$ is *connected*, it has a homotopy inverse, and for monoids M with a homotopy inverse there is a weak equivalence $M \simeq \Omega BM$; see Segal [18] for proofs of these statements. The Dold-Thom theorem then yields the weak equivalence of spectra $H\mathbb{Z} \wedge X \simeq \mathrm{Sp}(\mathrm{Sym}^\infty X)$.

Theorem 8.4. *If Γ is an infinite discrete group such that $H_*(\mathrm{Irr}_n^+(\Gamma); \mathbb{Z}) = 0$ for all $n \geqslant 0$ and all $* > k$, then $\pi_*\mathrm{Sp}(\mathrm{Rep}(\Gamma)) = 0$ for $* > k$, and the Bott map*

$$\beta_* \colon \pi_* K^{\mathrm{def}}(\Gamma) \longrightarrow \pi_{*+2} K^{\mathrm{def}}(\Gamma)$$

is an isomorphism for $* > k - 2$, and injective for $* = k - 2$.

Proof. By Theorem 8.2, it suffices to show that $\pi_* R^{\mathrm{def}}(\Gamma) = 0$ for $* > k$. By Theorem 8.3, we have

(13) $$\pi_* R^{\mathrm{def}}(\Gamma) = \operatorname*{colim}_n \pi_*\mathrm{Sp}(\mathrm{Rep}(\Gamma, n)),$$

and there are long exact sequences

$$\cdots \longrightarrow \pi_* H\mathbb{Z} \wedge \mathrm{Irr}_{n-1}^+(\Gamma) \longrightarrow \pi_*\mathrm{Sp}(\mathrm{Rep}(\Gamma, n-1)) \longrightarrow \pi_*\mathrm{Sp}(\mathrm{Rep}(\Gamma, n))$$

$$\longrightarrow \pi_* H\mathbb{Z} \wedge \mathrm{Irr}_n^+(\Gamma) \longrightarrow \cdots .$$

As noted above, $\pi_* H\mathbb{Z} \wedge \mathrm{Irr}_n^+(\Gamma) \cong \widetilde{H}_*(\mathrm{Irr}_n^+; \mathbb{Z})$, so these groups are zero for $* > k$. Hence for $* > k$, the maps

$$\pi_*\mathrm{Sp}(\mathrm{Rep}(\Gamma, n-1)) \longrightarrow \pi_*\mathrm{Sp}(\mathrm{Rep}(\Gamma, n))$$

are isomorphisms for every n. Since $\mathrm{Rep}(\Gamma, 0)$ consists of trivial representations only, $\mathrm{Sp}(\mathrm{Rep}(\Gamma, 0)) = \mathrm{Sp}(\mathbb{N}) = H\mathbb{Z}$, and $\pi_*\mathrm{Sp}(\mathrm{Rep}(\Gamma, 0)) = 0$ for $* > 0$. We now conclude from Theorem 8.3 that for $* > k$, $\pi_*\mathrm{Sp}(\mathrm{Rep}(\Gamma, n)) = 0$ for all n, and now (13) completes the proof. $\square$

Recall from the introduction that when Γ is crystallographic and torsion-free, k is precisely the dimension of the classifying space $\mathbb{R}^k/\Gamma$ of Γ, which is flat manifold (covered by the torus $\mathbb{R}^k/A$). So when $\mathbb{R}^k/\Gamma$ is orientable, we have shown that $K^{\mathrm{def}}(\Gamma)$ is periodic above the (rational) cohomological dimension of Γ, minus 2.

A standard transfer argument shows that in general, $H^*(\Gamma; \mathbb{Q}) = 0$ for $* > k$, but the rational cohomological dimension of Γ could be less than k. For instance, when Γ is crystallographic and torsion-free but $\mathbb{R}^k/\Gamma$ is *non*-orientable, then $\mathrm{qcd}(\Gamma)$ is strictly less than k.

In the author's previous work on surface groups [14], it was shown that if Γ is a the fundamental group of a product of aspherical surfaces, then the zeroth space of the spectrum $\mathrm{Sp}(\mathrm{Rep}(\Gamma))$ is weakly equivalent to the *stable moduli space*

$$\operatorname*{colim}_n \mathrm{Hom}(\Gamma, U(n))/U(n) \cong \mathrm{Hom}(\Gamma, U)/U.$$

This relies on the following fact: for each representation $\rho \colon \Gamma \to U(n)$, there exists a representation $\psi \colon \Gamma \to U(m)$ (for some m) such that $\rho \oplus \psi$ lies in the connected component of the trivial representation. We call the monoid $\mathrm{Rep}(\Gamma)$ *stably group-like* when it has this property.

Question 8.5. *For which crystallographic groups Γ is $\mathrm{Rep}(\Gamma)$ stably group-like?*

In the next section, we offer one interesting class of crystallographic groups for which $\mathrm{Rep}(\Gamma)$ *is* stably group-like.

9. The stable moduli space

In this section, we will consider what we call *flat iterated torus bundles*. This class of space forms $\mathcal{T}$ is constructed as follows. First, $\mathcal{T}_0$ denotes the class of (finite rank) tori. We recursively define $\mathcal{T}_i$ to be the class consisting of all space forms E which (geometrically) fiber over a space form in $\mathcal{T}_{i-1}$ with flat tori as fibers. Note that since space forms are determined up to affine equivalence by the isomorphism type of their fundamental group (Wolf [21, Theorem 3.3.1]), if $E \in \mathcal{T}_i$ then any space form with fundamental group isomorphic to $\pi_1 E$ also lies in $\mathcal{T}_i$. The class $\mathcal{T}$ is simply the union of the $\mathcal{T}_i$. The notion of geometric fibering is defined in Ratcliffe and Tschantz [16], and is a bit stronger than the topological notion of a fiber bundle. For our purposes, the important point will be that the fundamental groups of space forms in $\mathcal{T}$ admit particularly nice (recursive) decompositions.

Say $\Gamma = \pi_1 E$ for some $E \in \mathcal{T}_i$. Then there exists a (geometric) fiber bundle

$$(S^1)^l \longrightarrow E \longrightarrow E_{i-1}$$

with $E_{i-1} \in \mathcal{T}_{i-1}$, and from the long exact sequence in homotopy we see that $\pi_1(S^1)^l \cong \mathbb{Z}^l$ is normal in Γ, with quotient $\pi_1(E_{i-1})$. Ratcliffe and Tschantz show [16, §7, Lemma 5] that $\pi_1(S^1)^l$ lies inside the translation subgroup of Γ.

With this in mind, we define a class of groups $\mathcal{Z}$ as follows. First, $\mathcal{Z}_0$ is the class of finitely generated free abelian groups. We recursively define $\mathcal{Z}_i$ to be the class of *torsion-free* crystallographic groups Γ that sit in extensions

$$(14) \qquad\qquad\qquad \mathbb{Z}^l \longrightarrow \Gamma \longrightarrow \Gamma_{i-1},$$

with $\Gamma_{i-1} \in \mathcal{Z}_{i-1}$ and the image of $\mathbb{Z}^l$ contained in the translation subgroup of Γ. We let $\mathcal{Z} = \bigcup_i \mathcal{Z}_i$.

Recall here that if Γ is an abstract crystallographic group (i.e. Γ is isomorphic to some discrete, cocompact subgroup of the isometries of Euclidean space), then the translation sugroup of Γ is well-defined (Lemma 2.1).

Proposition 9.1. *The class $\mathcal{Z}$ is precisely the class of groups isomorphic to the fundamental group of some $E \in \mathcal{T}$.*

Proof. The above discussion shows, inductively, that if $E \in \mathcal{T}_i$ then $\pi_1 E \in \mathcal{Z}_i$. Conversely, we will show that for all i, each $\Gamma \in \mathcal{Z}_i$ is isomorphic to $\pi_1 E$ for some $E \in \mathcal{T}_i$. For $i = 0$, this is immediate from the definitions, so assume the statement for $i - 1$ and consider $\Gamma \in Z_i$. Then there exists an extension

$$\mathbb{Z}^l \longrightarrow \Gamma \longrightarrow \Gamma_{i-1},$$

with $\Gamma_{i-1} \in \mathcal{Z}_{i-1}$ and the image N of $\mathbb{Z}^l$ contained in the translation subgroup of Γ. If we consider the action of Γ on $\mathbb{R}^n$ by isometries, then each $x \in N$ acts via a translation

$$\vec{\mathbf{v}} \mapsto \vec{\mathbf{v}} + a_x$$

for some $a_x \in \mathbb{R}^n$. Set $V = \mathrm{Span}(\{a_x \,|\, x \in N\})$. Then the results of [16, §3, 6] tell us that $\Gamma/N \cong \Gamma_{i-1}$ acts effectively on $\mathbb{R}^n/V$ by isometries, and the quotient $(\mathbb{R}^n/V)/\Gamma_{i-1}$ is a (compact) space form. Moreover, the space form $\mathbb{R}^n/\Gamma$ geometrically fibers over $(\mathbb{R}^n/V)/\Gamma_{i-1}$ with fiber the flat torus $V/N \cong (S^1)^l$. Since $\Gamma_{i-1} \in \mathcal{Z}_{i-1}$, we know that $\Gamma_{i-1} = \pi_1 E_{i-1}$ for some space form $E_{i-1} \in \mathcal{T}_{i-1}$. As noted above, since $(\mathbb{R}^n/V)/\Gamma_{i-1}$ is a space form with fundamental group Γ_{i-1}, we have $(\mathbb{R}^n/V)/\Gamma_{i-1} \in \mathcal{T}_{i-1}$ as well. By definition of $\mathcal{T}_i$, we now see that $E \in \mathcal{T}_i$. $\square$

Our goal is to study the stable moduli space $\mathcal{M}_{\text{flat}}(E)$ of flat unitary connections over a flat iterated torus bundle E. As discussed in the introduction, the holonomy representation associated to a flat connection gives rise to a homeomorphism

$$\mathcal{M}_{\text{flat}}(E) \xrightarrow{\cong} \underset{n}{\text{colim}} \, \text{Hom}(\pi_1, U(n))/U(n) \cong \text{Hom}(\pi_1 E, U)/U.$$

Theorem 9.2. *Let E be a flat iterated torus bundle, and assume that the point group of $\pi_1 E$ is abelian. Then the homotopy groups of $\mathcal{M}_{flat}(E)$ vanish above the dimension of E. Moreover, the stable moduli space is homotopy equivalent to a finite product of Eilenberg–MacLane spaces:*

$$\mathcal{M}_{flat}(E) \simeq \prod_{i=0}^{\dim(E)} K\left(\pi_i(\mathcal{M}_{flat}(E)), i\right).$$

Remark 9.3. *Note that for flat torus bundles over tori, i.e. spaces $E \in \mathcal{T}_1$, the fundamental group $\pi_1 E$ always has abelian point group, since the point group is a quotient of the fundamental group of the base torus.*

Further bundles to which Theorem 9.2 applies may be constructed as follows. Take a group $\Gamma_{i-1} \in \mathcal{Z}_{i-1}$ with abelian *point group Q, and consider a semi-direct product $\Gamma = \mathbb{Z}^l \rtimes \Gamma_{i-1}$ in which Γ_{i-1} acts on $\mathbb{Z}^l$ via a representation $\Gamma_{i-1} \to Q \to \mathbb{Z}^l$ (which need not be faithful on Q). This gives a (split) extension*

$$\mathbb{Z}^l \longrightarrow \Gamma \longrightarrow \Gamma_{i-1}.$$

Let $A \lhd \Gamma_{i-1}$ denote the subgroup of translations. Then in the semi-direct product Γ, we see that A acts trivially on $\mathbb{Z}^l$. Hence these subgroups generate a free abelian subgroup $\mathbb{Z}^l \times A$ inside of Γ with quotient isomorphic to Q. The action of Q on $\mathbb{Z}^l \times A$ is simply the diagonal action, which is faithful since Q acts faithfully on A. It now follows from Ratcliffe [15, Theorem 7.5.5] that Γ is a crystallographic group, with translation subgroup $\mathbb{Z}^l \times A$. Since the semi-direct product of two torsion-free groups is always torsion-free, we find that $\Gamma \in \mathcal{Z}_i$, and $\mathbb{R}^{l+\text{rk}(A)}/\Gamma \in \mathcal{T}_i$.

We will prove Theorem 9.2 result by showing that for any $E \in \mathcal{T}$, $\mathbb{Z} \times \mathcal{M}_{\text{flat}}(E)$ is weakly equivalent to the zeroth space of the Ω–spectrum $\text{Sp}(\text{Rep}(\pi_1 E))$. This relies on the following general fact, proven in Ramras [13, Section 6].

Lemma 9.4. *Let Γ be a finitely generated discrete group, and assume that for all $\rho \colon \Gamma \to U(n)$ there exists $\psi \colon \Gamma \to U(m)$ such that $\rho \oplus \psi \in \text{Hom}(\Gamma, U(n + m))$ lies in the connected component of the trivial representation. Then then the zeroth space of the spectrum associated to the monoid $\text{Rep}(\Gamma)$ is weakly equivalent to $\mathbb{Z} \times \text{Hom}(\Gamma, U)/U$.*

We call $\text{Rep}(\Gamma)$ *stably-grouplike* when the hypotheses of Lemma 9.4 are satisfied. We will show that the groups in $\mathcal{Z}$ satisfy the even stronger condition that for each representation ρ, there exists an integer $m > 0$ such that the m–fold block sum of ρ with itself, denoted $m\rho$, lies in the connected component of the trivial representation.

Lemma 9.5. *Let Γ be a group in $\mathcal{Z}$, and let $A \cong \mathbb{Z}^k \leqslant \Gamma$ be the subgroup of translations. Let $I_n \in \text{Hom}(A, U(n))$ denote the trivial representation. Then for some $p > 0$, the p–fold block sum $p\text{Ind}_A^\Gamma(I_n)$ lies in the connected component of the trivial representation.*

Proof. We prove the result for each class $\mathcal{Z}_i$, by induction on i. When $i = 0$, Γ is free abelian of finite rank. The fact that commuting unitary matrices are simultaneously diagonalizable shows that $\mathrm{Hom}(\mathbb{Z}^r, U(m))$ is connected (for any r and m). We now assume the result for groups in $\mathcal{Z}_{i-1}$ and consider some $\Gamma \in \mathcal{Z}_i$. Consider an extension

$$\mathbb{Z}^l \longrightarrow \Gamma \longrightarrow \Gamma_{i-1}$$

of the form (14). Since $A \lhd \Gamma$, the representation $\mathrm{Ind}_A^\Gamma(I_n)$ is trivial on A, and also on $\mathbb{Z}^l \leqslant A$. We may now view $\mathrm{Ind}_A^\Gamma(I_n)$ as a unitary representation of the quotient group $\Gamma/\mathbb{Z}^l \cong \Gamma_{i-1}$. Since $\Gamma_{i-1} \in \mathcal{Z}_{i-1}$, by our induction hypothesis we know that there exists an integer m such that $p\mathrm{Ind}_A^\Gamma(I_n)$ lies in the connected component of the trivial representation in $\mathrm{Hom}(\Gamma_{i-1}, U(pn[\Gamma : A]))$. This yields a path in $\mathrm{Hom}(\Gamma, U(pn[\Gamma : A]))$ from $m\mathrm{Ind}_A^\Gamma(I_n)$ to the trivial representation. $\square$

Proposition 9.6. *For any group $\Gamma \in \mathcal{Z}$ and any representation $\rho : \Gamma \to U(n)$, there exists an integer $d > 0$ such that $d\rho$ lies in the connected component of the trivial representation in $\mathrm{Hom}(\Gamma, U(nm))$. In particular, the monoid $\mathrm{Rep}(\Gamma)$ is stably-grouplike.*

Proof. Again, we prove the result for each $\mathcal{Z}_i$ by induction on i, and the base case follows as in Lemma 9.5. We will use the notation $\psi \simeq \psi'$ to mean that there exists a path connecting the representations ψ and ψ'. Note that since $U(n)$ is connected, any two isomorphic representations are connected by a path: $\psi \cong \psi' \implies \psi \simeq \psi'$.

Say $\Gamma \in \mathcal{Z}_i$. Let $\mathbb{Z}^l \leqslant A \leqslant \Gamma$ be as in Lemma 9.5. By Lemma 9.5, $r\mathrm{Ind}_A^\Gamma(1) \simeq I_{r[\Gamma:A]}$ for some r (where 1 denotes the trivial 1–dimensional representation). Now,

$$r[\Gamma : A]\rho \cong \rho \otimes I_{r[\Gamma:A]} \simeq \rho \otimes \left(r\mathrm{Ind}_A^\Gamma(1) \right)$$
$$\cong \rho \otimes \mathrm{Ind}_A^\Gamma(I_r) \cong \mathrm{Ind}_A^\Gamma\left(\mathrm{Res}_A^\Gamma(\rho) \otimes I_r \right) \cong \mathrm{Ind}_A^\Gamma\left(r\mathrm{Res}_A^\Gamma(\rho) \right),$$

where we have used the Projection Formula (Lemma 5.6).

Since $r\mathrm{Res}_A^\Gamma(\rho)$ is a representation of the free abelian group A, we know that $qr\mathrm{Res}_A^\Gamma(\rho) \simeq I_{qrn}$ for some q. Let $\psi_t \colon [0, 1] \to \mathrm{Hom}(A, U(n))$ denote a continuous path with $\psi_0 = qr\mathrm{Res}_A^\Gamma(\rho)$ and $\psi_1 = I_{qrn}$. Then $\mathrm{Ind}_A^\Gamma(\psi_t)$ is a continuous path

$$q\mathrm{Ind}_A^\Gamma\left(r\mathrm{Res}_A^\Gamma(\rho) \right) \cong \mathrm{Ind}_A^\Gamma\left(qr\mathrm{Res}_A^\Gamma(\rho) \right) \simeq \mathrm{Ind}_A^\Gamma(I_{qrn}).$$

By Lemma 9.5, there exists s such that

$$s\mathrm{Ind}_A^\Gamma(I_{qrn})) \simeq I_{s[\Gamma:A]qrn}.$$

Combining the various paths displayed above yields a path from $sqr[\Gamma : A]\rho$ to the trivial representation. $\square$

Proof of Theorem 9.2. By Proposition 9.6 and Lemma 9.4, there is a weak equivalence $\mathbb{Z} \times \mathcal{M}_{\mathrm{flat}}(E) \simeq \Omega B\mathrm{Rep}(\pi_1 E)$. The homotopy groups of an Ω–spectrum agree with those of its zeroth space, and $\pi_* \mathrm{Sp}(\mathrm{Rep}(\pi_1 E)) = 0$ for $* > \dim(E)$ by Theorem 8.4. To see that $\mathcal{M}_{\mathrm{flat}}(E)$ has the homotopy type of a product of Eilenberg-MacLane spaces, one uses the fact that $\Omega B\mathrm{Rep}(\Gamma)$ is a topological abelian monoid. Note that triangulations of $\mathrm{Hom}(\Gamma, U(n))/U(n)$ give the stable moduli space $\mathcal{M}_{\mathrm{flat}}(E)$ the structure of an Ind-CW complex, so $\mathcal{M}_{\mathrm{flat}}(E)$ has the homotopy type of a CW-complex. See [13, Corollary 6.4] and [14, Lemma 5.7] for details. $\square$

We end this section by noting that *rational* vanishing results can be obtained without the assumption that the point group of Γ is abelian. This is essentially a representation-theoretical version of the statement that if a space X has a finite cover $Y \to X$ with $H^*(Y; \mathbb{Q}) = 0$ for $* > k$, then $H^*(X; \mathbb{Q}) = 0$ for $* > k$ as well. The proof of this cohomological statement is a simple application of the transfer; for the representation theoretical version induction maps will play the role of the transfer.

Proposition 9.7. *Let E be a flat iterated torus bundle. Then for $* > \dim(E)$, we have $\pi_* \mathcal{M}_{flat}(E) \otimes \mathbb{Q} = 0$, and consequently the rationalized Bott map*

$$\beta_*^{\mathbb{Q}} \colon K^{\mathrm{def}}(\pi_1 E) \otimes \mathbb{Q} \longrightarrow K^{\mathrm{def}}(\pi_1 E) \otimes \mathbb{Q}$$

is an isomorphism for $* > \dim(E) - 2$.

Proof. This is essentially a parametrized version of the proof of Proposition 9.6, so we will only sketch the argument. We will show that for any map $\rho \colon S^k \to \mathcal{M}_{\mathrm{flat}}(\pi_1 E)$, $k > \dim(E)$, the map $d\rho$ is nullhomotopic for some $d > 0$. Note here that $d\rho$ may be interpreted either as the d–fold block-sum of ρ, or as the d–fold sum of ρ with itself in $\pi_k \mathcal{M}_{\mathrm{flat}}(\pi_1 E)$. These classes are homotopic by the arguments in [12].

Consider the map $\rho \otimes \mathrm{Ind}_A^{\pi_1 E}(1) = \mathrm{Ind}_A^{\pi_1 E}(\mathrm{Res}_A^{\pi_1 E}(\rho))$ where $A \subset \pi_1 E$ is the translation subgroup. Since $\pi_* \mathcal{M}_{\mathrm{flat}}((S^1)^{\mathrm{rk}(A)}) = 0$ for $* > \mathrm{rk}(A) = \dim(E)$, we know that $\mathrm{Res}_A^{\pi_1 E}(\rho)$ is nullhomotopic, so $\rho \otimes \mathrm{Ind}_A^{\pi_1 E}(1)$ is nullhomotopic as well. By Lemma 9.5, we know that $t\mathrm{Ind}_A^{\pi_1 E}(1) \simeq I_{t[\Gamma:A]}$ for some $t > 0$. Tensoring this path with ρ shows that the nullhomotopic map $t\rho \otimes \mathrm{Ind}_A^{\pi_1 E}(1)$ is homotopic to $\rho \otimes I_{t[\Gamma:A]} \cong t[\Gamma:A]\rho$ $\qquad \square$

10. The Klein bottle

We end with an example, showing that when Γ is a torsion-free crystallographic group whose classifying space is non-orientable, $\mathrm{Irr}_n^+(\Gamma)$ can still have dimension equal to the rank of the translation subgroup of Γ. The smallest such crystallographic group is the fundamental group of the Klein bottle,

$$K = \langle c, d \,|\, cdc^{-1} = d^{-1} \rangle.$$

The subgroup of translations in K is generated by the commuting elements c^2 and d, so by Theorem 7.5 $\mathrm{Irr}_2^+(K)$ is at most two-dimensional.

Claim. *There is a homeomorphism $\mathrm{Irr}_2^+(K) \cong (S^1 \times S^1)/(S^1 \times \{1\})$.*

We sketch the proof. Let $\rho \colon K \to U(2)$ be an irreducible representation. Then ρ is determined by the two unitary matrices $C = \rho(c)$ and $D = \rho(d)$. Up to conjugacy in $U(2)$, we may assume that D is diagonal, and since D is conjugate to its inverse, the eigenvalues of D must be λ and λ^{-1} for some $\lambda \in S^1 \setminus \{\pm 1\}$ (if both eigenvalues lie in the set $\{\pm 1\}$, then ρ is reducible). Since C interchanges the eigenspaces of D, we see that

$$C = \begin{bmatrix} 0 & c_1 \\ c_2 & 0 \end{bmatrix}$$

for some $c_1, c_2 \in S^1$, and up to conjugacy we may assume that $c_1 = 1$. So every irreducible $U(2)$–representation of K is conjugate to one of the form

$$(15) \qquad \left(\begin{bmatrix} 0 & 1 \\ c & 0 \end{bmatrix}, \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \right),$$

with $c \in S^1$ and $\theta \in (0, 2\pi)$. The only redundancy in this collection is that conjugating (15) by $\begin{bmatrix} 0 & 1 \\ c & 0 \end{bmatrix}$ leaves the first coordinate fixed, but interchanges $e^{i\theta}$ and $e^{-i\theta}$. We now see that all irreducible representations have the form (15) with $c \in S^1$ and $\theta \in (0, \pi)$, and the one-point compactification $\mathrm{Irr}_2^+(K)$ is obtained by collapsing $S^1 \times \{0, 1\}$ to a point inside $S^1 \times [0, \pi]$.

Although the second homology group of $\mathrm{Irr}_2^+(K)$ is non-zero, an explicit analysis of the boundary maps for the sequences in Theorem 8.3 shows that $\pi_2 R^{\mathrm{def}}(K) = 0$. In fact, one may calculate $\pi_* R^{\mathrm{def}}(K)$ and $\pi_* K^{\mathrm{def}}(K)$ completely using Theorems 8.3 and 8.2, reproducing the calculations in [13]. The key is that the boundary maps for the cofiber sequence in Theorem 8.3 can be made quite explicit. These cofiber sequences result from the fact that the square of spectra

$$
\begin{array}{ccc}
\mathrm{Sp}\left(\mathrm{Sym}^\infty \left(\mathrm{Sum}(\Gamma, U(n))/U(n)\right)_+\right) & \longrightarrow & \mathrm{Sp}\left(\mathrm{Sym}^\infty \left(\mathrm{Hom}(\Gamma, U(n))/U(n)\right)_+\right) \\
\downarrow & & \downarrow \\
\mathrm{SpRep}_{n-1}(\Gamma) & \longrightarrow & \mathrm{SpRep}_n(\Gamma)
\end{array}
$$

is a homotopy pushout square, meaning that the induced map between the homotopy cofibers of this square is a weak equivalence (see Lawson [9, Section 2]). By the Dold-Thom Theorem, the homotopy cofiber of the first row is simply $Sp\left(\mathrm{Irr}_n^+(\Gamma)\right)$, and the boundary map for the top sequence agrees with the boundary map in homology for the cofiber sequence

$$(16) \qquad \mathrm{Sum}(\Gamma, U(n))/U(n) \longrightarrow \mathrm{Hom}(\Gamma, U(n))/U(n) \longrightarrow \mathrm{Irr}_n^+(\Gamma).$$

Next, the inclusion $\mathrm{Sum}(\Gamma, U(n))/U(n) \hookrightarrow \mathrm{Rep}_{n-1}(\Gamma)$ induces a map of monoids

$$\mathrm{Sym}^\infty \left(\mathrm{Sum}(\Gamma, U(n))/U(n)\right)_+ \longrightarrow \mathrm{Rep}_{n-1}(\Gamma),$$

and a map of spectra

$$(17) \qquad \mathrm{Sp}\left(\mathrm{Sym}^\infty \left(\mathrm{Sum}(\Gamma, U(n))/U(n)\right)_+\right) \longrightarrow \mathrm{SpRep}_{n-1}(\Gamma).$$

The boundary map for homotopy cofiber sequence in Theorem 8.3 is formed by composing the homological boundary map for the sequence (16) with the map on homotopy induced by (17), using the Dold-Thom Theorem to identify the target and source of these maps. For the Klein bottle, the homological boundary maps are easily computed, and the map $\mathrm{Sym}^\infty \left(\mathrm{Sum}(K, U(2))/U(2)\right)_+ \longrightarrow \mathrm{Rep}_1(K)$ can be understood using the homeomorphisms

$$\mathrm{Irr}_1(K) \cong S^1 \coprod S^1, \ \ \mathrm{Sum}(K, U(2))/U(2) \cong \mathrm{Sym}^2(\mathrm{Irr}_1(K)),$$

$$\mathrm{Rep}_1(K) \cong \mathrm{Sym}^\infty \mathrm{Irr}_1(K).$$

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New Mexico State University, Department of Mathematical Sciences, P.O. Box 30001, Department 3MB, Las Cruces, New Mexico 88003-8001

E-mail address: ramras@nmsu.edu

URL: http://www.math.nmsu.edu/~ramras/