BRUNNIAN BRAIDS

Abstract. We record some observation about Brunnian braids on the sphere, which lead to the conclusion that the the cyclic group $\mathbb{Z}/(k-1)\mathbb{Z}$ acts on

$$\mathrm{Brunn}_k(S^2)/j(\mathrm{Brunn}_k(D^2)) \cong \pi_{k-1}(S^2),$$

in k (potentially) different ways. Here j is induced by the natural inclusion of D^2 into S^2 as the upper hemisphere.

1. Some observations about Brunnian braids

Given a manifold X, let $F(X,k)$ denote the configuration space of k distinct, ordered points in X; that is

$$F(X,k) = \{(x_1,\ldots,x_k) \in X^k \,:\, x_i \neq x_j \text{ for } i \neq j\}.$$

The pure braid groups on k strands in X is defined as $P_k(X) = \pi_1 F(X,k)$; since any two configurations in X are connected by an isotopy, we will usually suppress a choice of basepoint configuration. We have deletion operators $\partial_l^X\colon F(X,k) \to F(X,k-1)$, $l = 1\ldots,k$, which are fibrations whose fibers are of the form $X \setminus c$, where $c \in F(X,k-1)$. The induced maps $d_l^X\colon P_k(X) \to P_{k-1}(X)$ correspond to deletion of strands. When $X = D^2$, we simply write ∂_l and d_l; when $X = S^2$ we write $\overline{\partial}_l$ and $\overline{d}_l$.

Viewing D^2 as the upper hemisphere in S^2 via the map $j(x,y) = (x,y,\sqrt{1-x^2-y^2})$, we obtain a (surjective) group homomorphism

$$j_k\colon P_k(D^2) \to P_k(S^2)$$

whose kernel will be denoted N_k (we will suppress k from the notation when no confusion is likely). Note that the deletion operators commute with these maps; that is, $d_l \circ j_k = j_{k-1} d_l$. By definition,

$$\mathrm{Brunn}_k(D^2) = \bigcap_{l=1}^{k} \ker(d_l)$$

and

$$\mathrm{Brunn}_k(S^2) = \bigcap_{l=1}^{k} \ker(\overline{d}_l).$$

Let $x_l = \frac{1}{2}e^{\pi i(l-1)/k}$, $l = 2, \dots, k$, and let $x_1 = 0$. The configuration $c = (x_1, \dots, x_k)$ will serve as our basepoint for $F(D^2, k)$, and $j(c)$ will be the basepoint of $F(S^2, k)$. The fiber of ∂_1 over $\partial_1(c)$ is homeomorphic to $D^2 \setminus \{x_2, \dots, x_k\}$, whose fundamental group (based at $x_1 = 0$) is free on generators γ_l represented by loops winding once counterclockwise around x_l ($l = 2, \dots, k$). We denote the corresponding braid in $P_k(D^2)$ by $p_{1,l}$; note that in this braid, the first strand runs once counterclockwise around the lth strand while the other strands remain straight.

We now consider the fibration sequence

$$D^2 \setminus \{x_2, \dots, x_k\} \longrightarrow F(D^2, k) \xrightarrow{\widehat{\partial_1}} F(D^2, k-1).$$

Since $\pi_2(F(D^2, k-1)) = 0$ (these configuration spaces are aspherical, as is well-known and can be proven by induction on n using these fibration sequences) the above fibration actually yields a short-exact sequence of fundamental groups

$$1 \longrightarrow \pi_1(D^2 \setminus \{x_2, \dots, x_k\}) \longrightarrow P_k(D^2) \xrightarrow{d_1} P_{k-1}(D^2) \longrightarrow 1. \tag{1}$$

In summary, we have the following fact.

Lemma 1.1. *The group* $\ker(d_1)$ *is freely generated by the braids* $p_{1,l}$.

Next, let $y_i = j(x_i) \in S^2$. Consider the commutative diagram

$$\begin{array}{ccccc} D^2 \setminus \{x_2, \dots, x_k\} & \longrightarrow & F(D^2, k) & \longrightarrow & F(D^2, k-1) \\ \downarrow & & \downarrow j_k & & \downarrow j_{k-1} \\ S^2 \setminus \{y_2, \dots, y_k\} & \longrightarrow & F(S^2, k) & \longrightarrow & F(S^2, k-1) \end{array}$$

in which the vertical maps are induced by j. The first row gives rise to the short exact sequence (1) on π_1. For $k \geqslant 4$, the second row also induces a short exact sequence on π_1, since $\pi_2(F(S^2, k-1)) = 0$ by a result of Fadell and Van Buskirk [2] (see also [1, Corollary 2.3]). Hence for $k \geqslant 4$, we obtain a diagram of groups

(2)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(D^2 \setminus \{x_2, \dots, x_k\}) & \xrightarrow{f} & P_k(D^2) & \xrightarrow{d_1} & P_{k-1}(D^2)) & \longrightarrow & 1 \\ & & \downarrow q & & \downarrow j_k & & \downarrow j_{k-1} & & \\ 1 & \longrightarrow & \pi_1(S^2 \setminus \{y_2, \dots, y_k\}) & \xrightarrow{g} & P_k(S^2) & \xrightarrow{\bar{d}_1} & P_{k-1}(S^2) & \longrightarrow & 1 \end{array}$$

with exact rows.

Letting $\overline{\gamma}_l = q(\gamma_l)$, the Seifert–Van Kampen Theorem yields

$$\pi_1(S^2 \setminus \{y_2, \dots, y_k\}) = \langle \overline{\gamma}_2, \dots, \overline{\gamma}_k \,|\, \overline{\gamma}_2\overline{\gamma}_3 \dots \overline{\gamma}_k \rangle;$$

in particular q is surjective with kernel the normal subgroup generated by $\gamma_2\gamma_3\dots\gamma_k$. Let $\overline{p}_{1,l}$ denote the braid $g(\overline{\gamma}_l) = j_k(p_{1,l})$. Rearranging the presentation also yields the following fact.

Lemma 1.2. *Assume that $k \geqslant 4$. Then $\ker(\overline{d}_1)$ is freely generated by any $k-2$ of the braids $\overline{p}_{1,l}$, $l = 2,\dots,k$.*

Corollary 1.3. *The groups $Brunn_k(D^2)$ and (for $k \geqslant 4$) $Brunn_k(S^2)$ are free.*

Indeed, they are subgroups of the free groups $\ker(d_1)$ and $\ker(\overline{d}_1)$, respectively. We note that, according to [1, Proposition 7.2.2], the group $\mathrm{Brunn}_4(S^2)$ is free on 5 generators, while $\mathrm{Brunn}_3(S^2) = P_3(S^2) \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Brunn}_2(S^2) = P_2(S^2) = \{1\}$.

The 9–Lemma states that a surjective map of exact sequences of groups induces an exact sequence of kernels. Applying this to Diagram 2, we obtain an inductive description of $N_k = \ker(j_k)$.

Lemma 1.4. *For $k \geqslant 4$, there is an exact sequence of groups*

$$1 \longrightarrow \ker(q) = \langle\langle \gamma_2\gamma_3\dots\gamma_k \rangle\rangle \longrightarrow N_k \longrightarrow N_{k-1} \longrightarrow 1.$$

Proposition 1.5. *Assume that $k \geqslant 4$, and let g be the map from (2). For $l = 2,\dots,k$,*

$$\ker(\overline{d}_1) \cap \ker(\overline{d}_l) = g\Big(\langle\langle \overline{\gamma}_l \rangle\rangle\Big) = \langle\langle \overline{p}_{1,l} \rangle\rangle, \tag{3}$$

and consequently (using injectivity of g)

$$Brunn_k(S^2) = \bigcap_{l=2}^{k} \langle\langle \overline{p}_{1,l} \rangle\rangle = \bigcap_{l=2}^{k} g\Big(\langle\langle \overline{\gamma}_l \rangle\rangle\Big) \cong \bigcap_{l=2}^{k} \langle\langle \overline{\gamma}_l \rangle\rangle.$$

Here $\langle\langle \overline{\gamma}_l \rangle\rangle$ denotes the normal subgroup of $\pi_1(S^2 \setminus (k-1))$ generated by $\overline{\gamma}_l$, and $\langle\langle \overline{p}_{1,l} \rangle\rangle$ is the normal subgroup of $P_k(S^2)$ generated by $\overline{p}_{1,l}$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} S^2 \setminus \{y_2,\dots,y_k\} & \xrightarrow{\ \widetilde{g}\ } & F(S^2,k) \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle \overline{\partial}_l} \\ S^2 \setminus \{y_2,\dots,\widehat{y_l},\dots,y_k\} & \longrightarrow & F(S^2,k-1), \end{array} \tag{4}$$

where the horizontal maps are inclusions of fibers of $\overline{\partial}_1$ (so $\widetilde{g}_* = g$). Since

$$\ker(\overline{d}_1) \cap \ker(\overline{d}_l) = \ker\Big(\overline{d}_l\colon\ \ker(\overline{d}_1) \longrightarrow P_{k-1}(S^2)\Big)$$

and the horizontal maps in (4) are injective on π_1, we see that

$$\ker(\overline{d}_1) \cap \ker(\overline{d}_l) = g(\ker(i_*)) = g\Big(\langle\langle \overline{\gamma}_l \rangle\rangle\Big).$$

It follows that $g\Big(\langle\langle \overline{\gamma}_l \rangle\rangle\Big) \lhd P_k(S^2)$, and since $g(\overline{\gamma}) = \overline{p}_{1,l}$ we have

$$\langle\langle \overline{p}_{1,l} \rangle\rangle \leqslant g\Big(\langle\langle \overline{\gamma}_l \rangle\rangle\Big),$$

and the reverse containment is immediate. $\square$

We note that the same argument gives the analogous result for the disk, without the restriction on k.

Proposition 1.6. *For $l = 2, \ldots, k$,*

$$\ker(d_1) \cap \ker(d_l) = f\Big(\langle\langle \gamma_l \rangle\rangle\Big) = \langle\langle p_{1,l} \rangle\rangle, \tag{5}$$

and consequently (since f is injective)

$$Brunn_k(D^2) = \bigcap_{l=2}^{k} f\Big(\langle\langle \gamma_l \rangle\rangle\Big) \cong \bigcap_{l=2}^{k} \langle\langle \gamma_l \rangle\rangle.$$

Here $\langle\langle \gamma_l \rangle\rangle$ denotes the normal subgroup of $\pi_1(D^2 \setminus (k-1))$ generated by γ_l.

Remark 1.7. *We note that Propositions 1.5 and 1.6 show that the normal subgroups of $P_k(S^2)$ and $P_k(D^2)$ generated by $\overline{p}_{1,l}$ and $p_{1,l}$ (respectively) are in fact the same as the normal subgroups of $\ker(\overline{d}_1)$ and $\ker(d_1)$ generated by these elements; that is, for any $\beta \in P_k(D^2)$ there exists $\beta' \in \ker(d_1)$ such that $\beta p_{1,l} \beta^{-1} = \beta' p_{1,l} (\beta')^{-1}$ and similarly for $P_k(S^2)$. Can this be seen directly from standard presentations of the braid group?*

The following observation will not be used in the next section, but seems worth recording anyhow.

Corollary 1.8. *For $k \geqslant 4$, there is an isomorphism*

$$\frac{Brunn_k(S^2)}{Brunn_k(D^2)} \cong \frac{\bigcap_{l=2}^{k} j\big(\langle\langle p_{1,l} \rangle\rangle\big)}{j\big(\bigcap_{l=2}^{k} \langle\langle p_{1,l} \rangle\rangle\big)}.$$

Proof. Since $j(p_{1,l}) = \overline{p}_{1,l}$ this follows from the Propositions 1.5 and 1.6, along with the fact that if $\pi\colon G \to H$ is a surjective group homomorphism, then for every $\gamma \in G$ we have $\pi(\langle\langle \gamma \rangle\rangle) = \langle\langle \pi(\gamma) \rangle\rangle$. $\square$

2. Operations on Brunnian braids

We have observed (Lemma 1.1) that $\ker(d_1)$ is free, with a relatively simple basis. In this section we use this fact to study the quotient groups $\mathrm{Brunn}_k(S^2)/j(\mathrm{Brunn}_k(D^2))$, which, by the main results of [1], are isomorphic to $\pi_{k-1}(S^2)$ when $k \geqslant 5$.

Proposition 2.1. *For $k \geqslant 4$, there is an injective group homomorphism*

$$\frac{Brunn_k(S^2)}{j(Brunn_k(D^2))} \overset{\phi}{\hookrightarrow} \frac{\ker(d_1)}{(N \cap \ker(d_1)) \cdot Brunn_k(D^2)} = \frac{\ker(d_1)}{(\langle\langle p_{1,2} \cdots p_{1,k} \rangle\rangle) \cdot Brunn_k(D^2)}.$$

Note here that $N \cap \ker(d_1) = \langle\langle p_{1,2} \cdots p_{1,k} \rangle\rangle$ by Lemma 1.4.

Remark 2.2. *One can see directly that in each of the quotients appearing in the Lemma, the subgroup is in fact* normal *in the larger group. To see that*

$$j(Brunn_k(D^2)) \lhd Brunn_k(S^2),$$

note that $Brunn_k(D^2) \lhd P_k(D^2)$ (as it is the intersection of the kernels of the homomorphisms d_l) and j is surjective. Also, $Brunn_k(D^2) \lhd P_k(D^2)$ implies that $Brunn_k(D^2)$ is also normal in the subgroup $\ker(d_1) \subset P_k(D^2)$, and since N and $\ker(d_1)$ are normal in $Brunn_k(D^2)$ so is their intersection. Thus $(N \cap \ker(d_1)) \cdot Brunn_k(D^2)$ is the product of two normal subgroups, hence normal.

Remark 2.3. *One might be able to say something interesting about $\pi_{k-1}(S^2)$ by analyzing torsion in the quotient group*

$$\frac{\ker(d_1)}{(N \cap \ker(d_1)) \cdot Brunn_k(D^2)}.$$

For instance, it would be very interesting to bound the size of abelian torsion subgroups in this quotient. Results of James, Moore, and Selick show that non-trivial elements in $\pi_{k-1}(S^2)$ with prime-power order always have either prime order or order 4 (see Section 8.5 of Berrick et al. [1]). So it would be interesting to understand elements of prime order in this quotient.

It's possible that the Proposition is somehow related to Theorem 1.3 of Berrick et al. [1].

Proof of Proposition 2.1. The necessary geometric facts all come from our analysis of Diagram (2). The remainder of the argument is completely formal.

To define ϕ, say $\beta \in \mathrm{Brunn}_k(S^2)$. We claim that $\beta \in j(\ker(d_1))$. In fact, chasing Diagram (2) shows that

$$\ker(\overline{d}_1) \subset j(\ker(d_1)).$$

We may now write $\beta = j(\alpha)$ for some $\alpha \in \ker(d_1)$, and we define

$$\phi([\beta]) = [\alpha].$$

To check that this gives a well-defined function, say $\beta' \in \mathrm{Brunn}_k(S^2)$ and $[\beta'] = [\beta] \in \dfrac{\mathrm{Brunn}_k(S^2)}{j(\mathrm{Brunn}_k(D^2))}$; then

$$\beta^{-1}\beta' \in j(\mathrm{Brunn}_k(D^2)). \tag{6}$$

We must show that if $\alpha' \in P_k(D^2)$ satisfies $j(\alpha') = \beta'$ and $d_1(\alpha') = 1$, then $[\alpha'] = [\alpha]$ in

$$\frac{\ker(d_1)}{(N \cap \ker(d_1)) \cdot \mathrm{Brunn}_k(D^2)},$$

or in other words that

$$\alpha^{-1}\alpha' \in (N \cap \ker(d_1)) \cdot \mathrm{Brunn}_k(D^2).$$

Lemma 2.4.

$$(N \cap \ker(d_1)) \cdot \mathit{Brunn}_k(D^2) = \ker(d_1) \cap j^{-1}\left(j(\mathit{Brunn}_k(D^2))\right).$$

Proof. For any group G and any normal subgroups $M, B, K \lhd G$ with $B \leq K$, an elementary computation shows that

$$(M \cap K)B = K \cap MB.$$

Now set $M = N$, $B = \mathrm{Brunn}_k(D^2)$, and $K = \ker(d_1)$. □

Returning to the proof of the Proposition, Lemma 2.4 (along with the fact that $\alpha^{-1}\alpha' \in \ker(d_1)$) tells us that to check that ϕ is a well-defined function it suffices to show that $j(\alpha^{-1}\alpha') \in j\left(\mathrm{Brunn}_k(D^2)\right)$. We have

$$j(\alpha^{-1}\alpha') = j(\alpha)^{-1}j(\alpha') = \beta^{-1}\beta',$$

and by (6), we have $\beta^{-1}\beta' \in j(\mathrm{Brunn}_k(D^2))$. This completes the proof that ϕ is well-defined.

To check that ϕ is a homomorphism, say $\beta, \beta' \in \mathrm{Brunn}_k(S^2)$. Choosing $\alpha, \alpha' \in \ker(d_1)$ with $j(\alpha) = \beta$ and $j(\alpha') = \beta'$, we have $\phi([\beta]) = [\alpha]$, $\phi([\beta']) = [\alpha']$. Since $j(\alpha\alpha') = \beta\beta'$ and $d_1(\alpha\alpha') = 1$, we have

$$\phi([\beta][\beta']) = [\alpha\alpha'] = [\alpha][\alpha'] = \phi([\beta])\phi([\beta']).$$

Lastly, we check that ϕ is injective. Say $\beta \in \mathrm{Brunn}_k(S^2)$ and $\phi([\beta]) = 1$; we must check that $\beta \in j\left(\mathrm{Brunn}_k(D^2)\right)$. By definition of ϕ, there exists $\alpha \in \ker(d_1)$ with $j(\alpha) = \beta$ and

$$\phi(\beta) = [\alpha] = 1 \in \frac{\ker(d_1)}{(N \cap \ker(d_1)) \cdot \mathrm{Brunn}_k(D^2)};$$

in other words $\alpha \in (N \cap \ker(d_1)) \cdot \mathrm{Brunn}_k(D^2)$. So there exist $n \in N \cap \ker(d_1)$ and $b \in \mathrm{Brunn}_k(D^2)$ such that $\alpha = nb$, and now

$$\beta = j(\alpha) = j(nb) = j(n)j(b) = j(b) \in j\left(\mathrm{Brunn}_k(D^2)\right).$$

□

As a result of Lemma 1.1, we have a natural action of the symmetric group $\Sigma_{\{2,\ldots,k\}}$ on $\ker(d_1)$, obtained by permuting the generators $p_{1,l}$, $l = 2, \ldots, k$. In the remainder of this section, we will study this action.

Lemma 2.5. *Assume that $k \geqslant 4$. Then the subgroup*

$$N \cap \ker(d_1) \lhd \ker(d_1)$$

is invariant under the action of the cyclic subgroup

$$\mathbb{Z}/(k-1)\mathbb{Z} \cong \langle (2\,3\,\ldots\,k) \rangle \leqslant \Sigma_{\{2,\ldots,k\}}.$$

Proof. Recall from Section 1 that we have a diagram of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(D^2 \setminus (k-1)) & \xrightarrow{f} & P_k(D^2) & \xrightarrow{d_1} & P_{k-1}(D^2)) & \longrightarrow & 1 \\ & & \downarrow q & & \downarrow j & & \downarrow j & & \\ 1 & \longrightarrow & \pi_1(S^2 \setminus (k-1)) & \xrightarrow{g} & P_k(S^2) & \xrightarrow{\overline{d}_1} & P_{k-1}(S^2) & \longrightarrow & 1 \end{array}$$

with exact rows, and q is surjective with kernel the normal subgroup generated by $\gamma_2\gamma_3\ldots\gamma_k$. This subgroup is preserved by the permutation $\sigma = (2\,3\,\ldots\,k)$, since $\gamma_2\ldots\gamma_k$ is conjugate to $\gamma_3\ldots\gamma_k\gamma_2$. Hence the action of $\mathbb{Z}/(k-1)\mathbb{Z}$ on $\pi_1(D^2 \setminus (k-1))$ descends to an action on $\pi_1(S^2 \setminus (k-1))$, making q a map of $\mathbb{Z}/(k-1)\mathbb{Z}$–modules.

Now, say $b \in N \cap \ker(d_1)$. We must show that $\sigma \cdot b \in N \cap \ker(d_1)$, and since σ acts on $\ker(d_1)$, we just need to check that $\sigma \cdot b \in N = \ker(j)$. Since $b \in \ker(d_1)$, we have $b = f(\gamma)$ for some $\gamma \in \pi_1(D^2 \setminus (k-1))$. Now,

$$gq(\gamma) = jf(\gamma) = j(b) = 1,$$

and injectivity of g implies that $q(\gamma) = 1$, so

$$q(\sigma \cdot \gamma) = \sigma \cdot q(\gamma) = \sigma \cdot 1 = 1.$$

Finally,

$$j(\sigma \cdot b) = j(\sigma \cdot f(\gamma)) = jf(\sigma \cdot \gamma) = gq(\sigma \cdot \gamma) = g(1) = 1.$$

□

Proposition 2.6. *For $k \geqslant 4$, there is an action of $\mathbb{Z}/(k-1)\mathbb{Z}$ on*

$$\frac{\ker(d_1)}{(N \cap \ker(d_1)) \cdot Brunn_k(D^2)},$$

and this action leaves the image of ϕ invariant.

Proof. By Lemma 2.5, the subgroup $N \cap \ker(d_1)$ is invariant under the action of $\mathbb{Z}/(k-1)\mathbb{Z}$ on $\ker(d_1)$, and the same is true of the subgroup $\mathrm{Brunn}_k(D^2)$ by Lemma 1.6. Since the action is through homomorphisms, the product of these two invariant subgroups is again invariant, and the action descends to an action on the quotient group. It remains to check that this action leaves the image of ϕ invariant.

Say $\sigma \in \mathbb{Z}/(k-1)\mathbb{Z}$, and let $\beta \in \mathrm{Brunn}_k(S^2)$ be given. We need to analyze $\sigma \cdot \phi([\beta])$. By construction of ϕ, there exists $\alpha \in \ker(d_1)$ such that $j(\alpha) = \beta$ and $\phi([\beta]) = [\alpha]$. To show that $\sigma \cdot \phi([\beta])$ lies in $\mathrm{Im}(\phi)$, it will suffice to check that $j(\sigma \cdot \alpha) \in \mathrm{Brunn}_k(S^2)$: indeed, if this is the case, then letting $\beta' = j(\sigma \cdot \alpha)$ we have $\phi([\beta']) = [\sigma \cdot \alpha]$ (note that $\sigma \cdot \alpha \in \ker(d_1)$ by definition of the action) and now

$$\sigma \cdot \phi([\beta]) = [\sigma \cdot \alpha] = \phi([\beta']) \in \mathrm{Im}(\phi).$$

By Proposition 1.5, to show that $j(\sigma \cdot \alpha) \in \mathrm{Brunn}_k(S^2)$ it will suffice to show that

$$j(\sigma \cdot \alpha) \in \bigcap_{l=2}^{k} g\Big(\langle\langle \overline{\gamma}_l \rangle\rangle\Big).$$

Since $\alpha \in \ker(d_1)$, we may write $\alpha = f(\widetilde{\alpha})$ for some $\widetilde{\alpha} \in \pi_1(D^2 \backslash (k-1))$. Now, $g \circ q(\widetilde{\alpha}) = \beta \in \mathrm{Brunn}_k(S^2)$, so applying Proposition 1.5 again, we have

$$g \circ q(\widetilde{\alpha}) \in \bigcap_{l=2}^{k} g\Big(\langle\langle \overline{\gamma}_l \rangle\rangle\Big),$$

and injectivity of g implies that in fact

$$q(\widetilde{\alpha}) \in \bigcap_{l=2}^{k} \langle\langle \overline{\gamma}_l \rangle\rangle. \tag{7}$$

By definition, $\sigma(\langle\langle \overline{\gamma}_l \rangle\rangle) \subset \langle\langle \overline{\gamma}_{\sigma(l)} \rangle$ for each l, so now

$$\sigma \cdot (q(\widetilde{\alpha})) \in \bigcap_{l=2}^{k} \langle\langle \overline{\gamma}_{\sigma(l)} \rangle\rangle = \bigcap_{l=2}^{k} \langle\langle \overline{\gamma}_l \rangle\rangle.$$

Finally,

$$j(\sigma\cdot\alpha)=j(\sigma\cdot f(\widetilde{\alpha}))=j\circ f(\sigma\cdot\widetilde{\alpha})=g\circ q(\sigma\cdot\widetilde{\alpha})=g(\sigma\cdot q(\widetilde{\alpha}))$$
$$\in g\left(\bigcap_{l=2}^{k}\langle\langle\overline{\gamma}_l\rangle\rangle\right)=\bigcap_{l=2}^{k}g\big(\langle\langle\overline{\gamma}_l\rangle\rangle\big).$$

□

We now have an action of $\mathbb{Z}/(k-1)\mathbb{Z}$ on $\pi_{k-1}S^2$ ($k\geqslant 5$). In fact, the whole argument could be repeated using d_j instead of d_1 ($j=2,3,\ldots,k$) so in fact we obtain an action of the k–fold free product of $\mathbb{Z}/(k-1)\mathbb{Z}$ on $\pi_{k-1}(S^2)$.

Question 2.7. *Is this action ever non-trivial? Note that when $k-1$ is even, we have the possibility that this action factors through the $\mathbb{Z}/2\mathbb{Z}$–action given by inversion in the abelian group $\pi_{k-1}(S^2)$. Based on the orders of the groups $\pi_{k-1}(S^2)$ (as listed on Wikipedia), the only $k-1\leqslant 21$ for which more interesting actions are possible are $k-1=6,10,12,14,18,20,21$. When $k-1=6,10,18$, all possible actions are inversion on some direct factors and trivial on others. For $k-1=12,14,20$ there is the possibility of an action permuting $\mathbb{Z}/2$ factors; for $k-1=12$ this is the only possibility and for $k=14,20$ there may be more complicated possible actions (the automorphism group in both cases contains Aut$(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z})$, which has order $3\cdot 2^6$). When $k-1=21$, again the automorphism group contains Aut$(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z})$, meaning that there is an automorphism of order 3 (giving a non-trivial action of $\mathbb{Z}/21\mathbb{Z}$).*

References

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