

HOMOTOPY INVARIANCE OF DEFORMATION K-THEORY

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ABSTRACT. We extend Carlsson's deformation K-theory to spaces equipped with an action by a discrete monoid, and show that this theory is homotopy invariant under (strong) equivariant homotopy equivalence and under taking products with free monoids. These properties follow from simple arguments analogous to the homotopy invariance of Weibel's homotopy K-theory [6].

1. INTRODUCTION

Deformation K-theory was originally defined by Carlsson [1] for a discrete group G . In this context, $K^{def}(G)$ is a connective Ω -spectrum, contravariantly functorial in G , built out of continuous families (deformations) of complex (unitary) representations. Although the bundle homotopy theorem guarantees that there are no non-trivial deformations of vector bundles, the action of a monoid on a bundle can be deformed non-trivially (when the base space is a point we are just considering deformations of representations). Given a space X together with an action of the monoid M on X , we will build a spectrum $K^{def}(X, M)$ out of continuous families of equivariant vector bundles over X (see Definition 2.3).

The present note explains two invariance properties of this functor. First, if (X, M) and (Y, N) are (strongly) equivariantly homotopy equivalent (Definition 3.4), then $K^{def}(X, M) \simeq K^{def}(Y, N)$ (Corollary 3.5). Next, if F is a free (abelian or non-abelian) monoid then given any space with monoid action (X, M) , the projection maps $(X, M \times F) \rightarrow (X, M)$ and $(X, M * F) \rightarrow (X, M)$ induce an equivalences on deformation K-theory (Corollary 3.6). (Here F acts trivially.) These results follow from a general fact about “homotopic” functors between categories of equivariant bundles.

In future work, ring and algebra structures will be considered in this context, using the recent results of Elmendorf and Mandell [2]

2. DEFINITIONS

We will consider topological spaces X equipped with an action by a discrete monoid M , and we will assume throughout that the identity $e \in M$ acts trivially. A morphism $(X, M) \rightarrow (Y, N)$ between spaces with monoid actions consists of a homomorphism (again preserving the identity) $\phi : M \rightarrow N$ and a map $f : X \rightarrow Y$ equivariant with respect to ϕ .

Definition 2.1. *Let (X, M) be a space with monoid action. The category of equivariant vector bundles over (X, M) is the category $Vect(X, M)$ with objects (E, p, α) where E is a (complex) vector bundle over X with projection $p : E \rightarrow X$, and α is a linear action of M on E making p equivariant. Morphisms in $Vect(X, M)$ are bundle maps (covering id_X) which are equivariant on the total spaces.*

The category $Vect_n(X, M)$ is defined to be $Vect(X \times \Delta^n, M)$ where Δ^n is the standard topological n -simplex, acted on trivially by M .

Remark 2.2. *Although the category of spaces with monoid action allows morphisms $(X, M) \rightarrow (X, M)$ which are not the identity on M , in the category $Vect(X, M)$, morphisms $E \rightarrow E'$ are always equivariant with respect to the identity map $M \rightarrow M$.*

Note that the categories $Vect_n(X, M)$ form a lax simplicial category, which we denote $Vect.(X, M)$. The structure maps are the functors induced by pull-back along the standard face and degeneracy maps in the cosimplicial space Δ^\cdot , and satisfy the simplicial identities only up to coherent natural transformations. Each category $Vect_n(X, M)$ admits direct sums (the direct sum of two equivariant bundles is the Whitney sum of the bundles acted on via the block sum of the action maps) and we in fact have a (lax) simplicial symmetric monoidal category. In order to define deformation K-theory, one must replace this lax functor with a strict functor. This may be done by the general machinery of Kleisli rectification (see [5]), or more concretely by replacing $Vect(X, M)$ with a subcategory $Vect'$ which is equivalent and strictly functorial.

We define $Vect'(X, M)$ to be the (small) full subcategory of the (large) category of all equivariant complex vector bundles E on X consisting of those E whose total space is, as a set, $X \times \mathbb{C}^n$ for some n . (This does not mean we are only considering trivial bundles; the topology on this set need not be the product topology.) We also assume that the projection map $E \rightarrow X$ is the natural map $X \times \mathbb{C}^n \rightarrow X$, and that the vector space structure on the fibres is the normal vector space structure on \mathbb{C}^n . (Such a bundle is completely determined by a

topology $T \subset \mathcal{P}(X \times \mathbb{C}^n)$ satisfying various hypotheses, such as local triviality.) Every vector bundle is isomorphic to such a bundle, by transport of structure along some fibrewise-linear set-map.

Within this subcategory we can define a functorial pull-back map. Given $(f, \phi) : (X, M) \rightarrow (Y, N)$ and an equivariant bundle $E = (Y \times \mathbb{C}^n, T)$ over Y (here T is the topology) we set $f^*E = (X \times \mathbb{C}^n, f^*T)$ where a subset U is in f^*T if and only if its image in the topological pull-back $X \times_Y (Y \times \mathbb{C}^n)$ (under the map sending (x, v) to $(x, f(x), v)$) is open in the product topology. (The action of M is defined in the obvious manner.) One easily checks that this is functorial; the main observation is that if we have a second map $(g, \psi) : (Z, P) \rightarrow (X, M)$ then the natural homeomorphism from $g^*f^*E = (Z \times \mathbb{C}^n, g^*f^*T)$ to $(f \circ g)^*E = (Z \times \mathbb{C}^n, (f \circ g)^*T)$ (coming from the universal property of pull-backs) is actually the identity as a set-map, and hence the two topologies coincide.

Note that Kleisli rectification replaces a category by a homotopy equivalent category, so by a result of Thomason [5] the two possible notions of deformation K-theory will coincide up to homotopy. From now on we will assume our functor $Vect$ is a strict functor to symmetric monoidal categories.

Definition 2.3. *The deformation K-theory of (X, M) , denoted $K^{def}(X, M)$, is defined to be the total spectrum of the simplicial Ω -spectrum associated to the simplicial symmetric monoidal category $Vect.(X, M)$. Note that $K^{def}(X, M)$ is a connective Ω -spectrum, contravariantly functorial in (X, M) .*

3. HOMOTOPIC BUNDLE FUNCTORS

The desired invariance results will be corollaries of a general result regarding functors between categories of equivariant bundles.

Definition 3.1. *Let (X, M) and (Y, N) be spaces with monoid action. An (additive) bundle functor from (Y, N) to (X, M) is a simplicial functor*

$$F : Vect.(Y, N) \rightarrow Vect.(X, M)$$

which is level-wise additive. We call two bundle functors $F, G : Vect.(Y, N) \rightarrow Vect.(X, M)$ homotopic if there is a bundle functor $H : Vect.(Y, N) \rightarrow Vect.(X \times I, M)$ (called a homotopy of bundle functors) which restricts to F and G (respectively) under the inclusions $X \times 0 \hookrightarrow X \times I$ and $X \times 1 \hookrightarrow X \times I$. (Here I is the unit interval, acted on trivially by M .)

Proposition 3.2. *Homotopic bundle functors induce homotopic maps on deformation K-theory spectra.*

Proof. Consider the diagram of simplicial categories

$$\text{Vect.}(Y, N) \longrightarrow \text{Vect.}(X \times I, M) \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{1} \end{array} \text{Vect.}(X, M) .$$

It will suffice to show that the functors denoted by 0 and 1, which are induced by the obvious restriction maps, are simplicially homotopic as functors between simplicial categories (in the sense of [4, p. 12]). In fact, the inclusion maps

$$(X \times \Delta^{\cdot}, M) \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{1} \end{array} (X \times I \times \Delta^{\cdot}; M) .$$

inducing these functors are cosimplicially homotopic: the standard triangulation maps $\phi_i : \Delta^{n+1} \rightarrow I \times \Delta^n$ (given, for example, in [3, p. 112]) satisfy the duals of the identities from [4] (this straightforward computation is left to the reader). \square

Remark 3.3. *The proof of Proposition 3.2 is similar to Weibel's proof that homotopy K-theory is a homotopy invariant functor on the category of rings [6]. It is not clear how to write down a cosimplicial homotopy between the map*

$$X \times I \times \Delta^{\cdot} \rightarrow X \times I \times \Delta^{\cdot}$$

sending $t \in I$ to 0 and the identity map. This would provide an alternate proof that the above maps 0 and 1 are homotopic, since they are each sections of the map $t \mapsto 0$. (Weibel's proof gives a simplicial homotopy between the analogous maps of simplicial rings).

We now explain the two invariance results which follow from Proposition 3.2.

Definition 3.4. *Let (X, M) and (Y, N) be spaces with monoid actions. A map $(f, \phi) : (X, M) \rightarrow (Y, N)$ is a strong equivariant homotopy equivalence if there exists a map $(g, \psi) : (Y, N) \rightarrow (X, M)$ together with equivariant homotopies $H : (X \times I, M) \rightarrow (X, M)$ from gf to id_X and $H' : (Y \times I, N) \rightarrow (Y, N)$ from fg to id_Y . (Here the actions of M and N on I are trivial).*

Since pull-back along an equivariant homotopy is a homotopy of bundle functors, we have:

Corollary 3.5. *If $H : (X \times I, M) \rightarrow (Y, N)$ is an equivariant homotopy (here M acts trivially on I) then the induced maps on deformation K-theory*

$$K^{def}(H_0), K^{def}(H_1) : K^{def}(Y, N) \rightarrow K^{def}(X, M)$$

are homotopic. Hence a strong equivariant homotopy equivalence $f : (X, M) \rightarrow (Y, N)$ induces a weak equivalence on deformation K-theory.

Next we discuss the effect in deformation K-theory of trivial actions by free monoids.

Corollary 3.6. *Let \mathbb{Z}_+ denote the monoid of non-negative integers, i.e. the free monoid on one generator. Let M be any monoid, and let $q : M \times \mathbb{Z}_+ \rightarrow M$ denote the projection. For any space with monoid action (X, M) , the projection map $(Id_x, q) : (X, M \times (\mathbb{Z}_+^n)) \rightarrow (X, M)$ induces a weak equivalence on deformation K-theory, with inverse induced by the inclusion map $i : (X, M) \rightarrow (X, M \times (\mathbb{Z}_+)^n)$. Here $(\mathbb{Z}_+)^n$ acts trivially on X .*

Proof. By induction it suffices to consider the case $n = 1$. Note that $qi = id_{(X, M)}$ and hence the induced bundle functor i^*q^* is the identity. Hence it will suffice to give a homotopy H between the bundle functors q^*i^* and $id_{(X, M \times \mathbb{Z}_+)}$.

First we define H on objects. Let $\alpha : M \rightarrow Map(X, X)$ denote the action, and say (E, p) is an equivariant vector bundle over $(X \times \Delta^n, M \times \mathbb{Z}_+)$, with action $\tilde{\alpha} : M \times \mathbb{Z}_+ \rightarrow End(E)$. We need to define a bundle $H_n(E) \in Vect_n(X \times I, M \times \mathbb{Z}_+)$ connecting (E, p) to $q^*i^*(E, p)$. As a vector bundle we simply take $H(E) = E \times I$ with the obvious projection onto $X \times I \times \Delta^n$. We need to define an action β of $M \times \mathbb{Z}_+$ on $E \times I$.

We will write \mathbb{Z}_+ additively, with generator $1 \in \mathbb{Z}_+$, and we will write M multiplicatively with identity $e \in M$. To give a map

$$\beta : M \times \mathbb{Z}_+ \longrightarrow End(E \times I)$$

we simply need to give a map $\phi : M \longrightarrow End(E \times I)$ and an element $A \in End(E \times I)$ such that $\phi(m)A = A\phi(m)$ for all $m \in M$. We define

$$\phi(m) = \tilde{\alpha}(m, 0) \times Id_I$$

and

$$A = t\tilde{\alpha}(e, 1) \times Id_I + (1 - t)Id_{E \times I};$$

note that $End(E \times I)$ is a vector space so this expression makes sense. The fact that $\phi(m)$ commutes with A for each $m \in M$ follows immediately from linearity of $\phi(m)$, together with the fact that $\tilde{\alpha}(m, 0)$ commutes with $\tilde{\alpha}(e, 1)$. Hence we obtain a well-defined action

$$\beta : M \times \mathbb{Z}_+ \longrightarrow End(E \times I),$$

which clearly makes p equivariant.

To define the functor H_n on morphisms, let $E \xrightarrow{f} E'$ be a morphism in $Vect_n(X, M \times \mathbb{Z}_+)$ and set $H_n(f) = f \times Id : E \times I \rightarrow E' \times I$. It is easily checked that this map is equivariant with respect to the above actions on $H_n(E) = E \times I$ and $H_n(E') = E' \times I$.

We now have a series of functors $H_n : Vect_n(X, M \times \mathbb{Z}_+) \rightarrow Vect_n(X \times I, M \times \mathbb{Z}_+)$, which are easily seen to be simplicial and additive. The restrictions at time 0 and 1 are the functors Id and q^*i^* (respectively), so the corollary now follows from Proposition 3.2. \square

Remark 3.7. *In order to make the functors H_n in the proof of Corollary 3.6 actually simplicial (when using the categories $\text{Vect}'(X, M)$ discussed above), we need to be careful about the definition of the bundle $E \times I$ (corresponding to a bundle $E \rightarrow Z$). If we are thinking of a bundle as a topology on the set $X \times \mathbb{C}^n$, then we can define $E \times I$ as simply the product topology on $X \times I \times \mathbb{C}^n$. It is easy to check that with this convention the H_n are simplicial (one sees that for any bundle $E \rightarrow X \times \Delta^n$, the natural homeomorphism*

$$((id_X \times \delta^i)^* E) \times I \longrightarrow (id_X \times \delta^i \times id_I)^*(E \times I)$$

is actually the identity as a set-map from $X \times \Delta^{n-1} \times I \times \mathbb{C}^n$ to itself.

The following result is proved by the same argument as in the proof of Corollary 3.6.

Corollary 3.8. *The following projection maps all induce weak equivalences on deformation K -theory, with inverses given by the obvious inclusion maps (here F_n is the free non-abelian monoid of n generators and $*$ denotes the free product of monoids):*

- $(X, M \times F_n) \rightarrow (X, M)$,
- $(X, M * F_n) \rightarrow (X, M)$,
- $(X, M * \mathbb{Z}_+^n) \rightarrow (X, M)$.

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