To derive the LES for good pairs $A \subset X$, we'll first need to understand the <u>relative</u> homology groups

$$H_k(X,A).$$

<u>Def</u>: For any spaces $A \subset X$, $C_*(X,A)$ is the chain cplx

$$C_n(X,A) := \frac{C_n X}{C_n A}$$

w/ bdry map induced by the universal prop. of the quotient map $C_n X \longrightarrow C_n X / C_n A$:

$$\begin{tikzcd} C_n A \arrow[r, "\partial"] \arrow[d, hook] & C_{n-1} A \arrow[d, hook] \\ C_n X \arrow[r, "\partial"] \arrow[d] & C_{n-1} X \arrow[d] \\ \frac{C_n X}{C_n A} \arrow[r, dashed, "\partial_{rel}"] & \frac{C_{n-1} X}{C_{n-1} A} \end{tikzcd}$$

<u>Note</u>: $\partial_{rel}^2 = 0$ b/c $\partial_{rel}^2([x] + C_n A) = \partial_{rel}([\partial x] + C_{n-1}A)$

$$= [\partial^2 x] + C_{n-1}A = 0.$$

<u>More generally</u>: If $(C_*, \partial_C) \xhookrightarrow{i} (D_*, \partial_D)$ is an inclusion of chain cplxs, i.e. i is a chain map & $C_n \xrightarrow{i} D_n$ is 1-1 $\forall n$, then D_n / C_n is a chain cplx w/ differential induced by ∂_C as above.

<u>Note</u>: The quotient maps $D_n \xrightarrow{q} D_n / C_n$ define a chain map.

Theorem: If $C \overset{i}{\hookrightarrow} D \overset{q}{\twoheadrightarrow} D/C$ is a SES of chain cplxes (i.e. is an inclusion & q is the quotient map) then $\exists$ a LES in homology of the form

$$\cdots \xrightarrow{q_*} H_{n+1}(D/C) \xrightarrow{\partial} H_n C \xrightarrow{i_*} H_n D \xrightarrow{q_*} H_n(D/C) \xrightarrow{\partial} H_{n-1}(C) \xrightarrow{i_*} \cdots$$

Moreover, a chain map $D \xrightarrow{f} D'$ w/ $f(C) \subset C'$ induces a comm. diag. of LES's

Note: Sometimes $H_n(D/C)$ is written $H_n(D,C)$

Pf (sketch): Each $x \in H_n(D/C)$ is rep'd by a class $\delta \in D_n$ w/ $d(\delta) \in C_{n-1}$. We define $\partial(x) = [d(\delta)] \in H_{n-1}C$. There are many things to check:

- Note that $d^2\delta = 0 \Rightarrow d(\delta)$ represents a cycle in D/C, so $[d\delta] \in H_{n-1}C$.
- If $x = [\delta']$ for some other $\delta' \in D_n$, then $\delta - \delta'$ is a bdry in D/C, i.e. $\delta - \delta' = d(\delta'') + c$ w/ $\delta'' \in D_{n+1}$, $c \in C_n$. Now
$$d(\delta - \delta') = d^2\delta'' + dc = dc, \text{ a bdry in } C.$$
so $[d\delta] = [d\delta']$ in $H_{n-1}C$
- δ is a homom. b/c if $x = [\delta]$, $y = [\delta']$ then $x+y = [\delta+\delta']$
$$\Rightarrow \partial(x+y) = [d(\delta+\delta')] = [d\delta + d\delta'] = [d\delta] + [d\delta'].$$
- Exactness of the seq. (see Hatcher).
- Commutativity of
$$\begin{array}{ccccccc} H_n C & \to & H_n D & \to & H_n D/C & \xrightarrow{\partial} & H_{n-1} C \\ \downarrow (f|_C)_* & & \downarrow f_* & & \downarrow \bar{f}_* & & \downarrow (f|_C)_* \\ H_n C' & \to & H_n D' & \to & H_{n-1} D'/C' & \xrightarrow{\partial} & H_{n-1} C' \end{array}$$

  is immediate except for the square involving ∂, but there we have

  $x = [\delta] \Rightarrow \bar{f}_*(x) = [f\delta]$

  $$(f|_C)_*(\partial x) = f([d\delta]) = [f(d\delta)] = [d(f\delta)] = \partial[\bar{f}_*(x)]$$

  f is a chain map

Excision:

Thm: Say $Z \subset A \subset X$ and $\overline{Z} \subset \mathrm{int}(A)$. Then $(X-Z, A-Z) \hookrightarrow (X,A)$ induces isom's $H_*(X-Z, A-Z) \xrightarrow{\cong} H_*(X,A)$ for all $n \geq 0$.

closure

Equivalently: If $X = A \cup B$, where $\mathrm{int}(A) \cup \mathrm{int}(B) = X$, then

$$H_*(B, A\cap B) \xrightarrow{\cong} H_*(X,A)$$

for $* \geq 0$. [Set $B = X - Z$ or $Z = X - B$ to go b/w the versions.]

Application: (Brouwer, 1910)

Thm: If $U \overset{\circ}{\subset} \mathbb{R}^n$ & $V \overset{\circ}{\subset} \mathbb{R}^m$ are homeomorphic, then $n = m$.

Pf: Study the local homology gps $H_k(U, U - \{x\})$ for pts $x \in U$ (and similarly for pts $y \in V$).

We can apply excision to the pair $(\mathbb{R}^n, \mathbb{R}^n - \{x\})$, w/ $Z = U^c$; note that $\mathbb{R}^n - U = \overline{\mathbb{R}^n - U} \subset \mathbb{R}^n - \{x\}$. We find:

$$H_k(\mathbb{R}^n, \mathbb{R}^n - x) \cong H_k(U, U - x).$$

Now the LES for $(\mathbb{R}^n, \mathbb{R}^n - x)$ gives

$$0 = H_k\mathbb{R}^n \longrightarrow H_k(\mathbb{R}^n, \mathbb{R}^n - x) \xrightarrow{\partial} H_{k-1}(\mathbb{R}^n - x) \longrightarrow H_{k-1}\mathbb{R}^n = 0$$

So ∂ is an isom.

$$H_k(\mathbb{R}^n, \mathbb{R}^n - x) \xrightarrow{\cong} H_k(\mathbb{R}^n - x) \cong H_{k-1} S^{n-1} \cong \begin{cases} \mathbb{Z}, & k = n, 0 \\ 0, & \text{else} \end{cases}$$

$\mathbb{R}^n - x \simeq$ unit sphere around x.

Now, a homeom. $f: U \xrightarrow{\cong} V$ would induce a homeom.

$$U - x \xrightarrow{\cong} V - f(x)$$

and hence an isom. $H_k(U, U-x) \xrightarrow{\cong} H_k(V, V-x)$

From our computation of these gps, we see that $n = m$ (since n is the only non-zero dim'n where $H_k(U, U-x) \neq 0$, and m is the only non-zero dim'n where $H_k(V, V-x) \neq 0$. $\square$

<u>The LES of a good pair</u>: For computations, the most important application of excision is:

$A \subseteq X$, $A \neq \emptyset$, and $A \subset V \overset{\circ}{\subseteq} X$ w/ $V \simeq A$

<u>Prop 2.22</u>: For good pairs (X, A), the quotient map $(X, A) \xrightarrow{q} (X/A, A/A)$ induces isom's $q_*: H_n(X, A) \xrightarrow{\cong} H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$ for all $n \geq 0$

A/A = point, so this follows from the LES of the pair.

<u>Lemma</u> (The 5-Lemma)

Say
$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$
is commutative and both rows are <u>exact</u>. Then if $\alpha, \beta, \delta, \varepsilon$ are isom's, so is γ.

In fact, to γ is onto so long as β & δ are onto & ε is 1-1, and γ is 1-1 so long as β & δ are 1-1 and α is onto.

<u>Pf</u>: By a "diagram chase". See Wikipedia.

<u>Ex</u>: If $(X, A) \to (Y, B)$ is a map of pairs & $X \to Y$, $A \to B$ are htpy equiv.'s, then $H_n(X, A) \to H_n(Y, B)$ is an isom $\forall n$. Indeed, we have a comm. diagram
$$\begin{tikzcd} H_n A \arrow[r] \arrow[d, "\cong"] & H_n X \arrow[r] \arrow[d, "\cong"] & H_n(X, A) \arrow[r] \arrow[d] & H_{n-1} A \arrow[r] \arrow[d, "\cong"] & H_{n-1} X \arrow[d, "\cong"] \\ H_n B \arrow[r] & H_n Y \arrow[r] & H_n(Y, B) \arrow[r] & H_{n-1} A \arrow[r] & H_{n-1} X \end{tikzcd}$$
& 5-Lemma applies.

Pf of 2.22: Analyze

By 5-lemma applied to the map of LES's

$$\begin{array}{ccccc} H_n(X,A) & \xrightarrow{\cong} & H_n(X,V) & \xrightarrow[\text{excision}]{\cong} & H_n(X-A, V-A) \\ \downarrow q_* & & \downarrow q_* & & \downarrow q_* \\ H_n(X/A, A/A) & \xrightarrow{\cong} & H_n(X/A, V/A) & \xrightarrow[\text{excision}]{\cong} & H_n(X/A - A/A, V/A - A/A) \end{array}$$

$V/A \simeq A/A$ so 5-lemma applies again

Proof of Excision:

Def: $C_n(A+B) \subset C_n(X)$ is the subgp gen'd by $C_n(A)$ & $C_n(B)$.

Note that $C_\cdot(A+B)$ is a sub-chain cplx of $C_n(X)$, b/c ∂ maps $C_\cdot(A)$ & $C_\cdot(B)$ into themselves.

Similarly, we can define $C_n(A+B, A) = C_n(A+B)/C_n(A)$. This gives a sub-chain cplx of $C_n(X,A)$. In fact,

$$(\star) \quad C_n(B, A\cap B) \xrightarrow{\cong} C_n(A+B, A)$$

b/c both are free abelian w/ basis $\{\sigma : \Delta^n \to B : \mathrm{Im}(\sigma) \not\subset A\}$.

The proof of excision is based on the following lemma:

Lemma: If $X = A \cup B$ as above, then

$$C_n(A+B, A) \longrightarrow C_n(X,A)$$

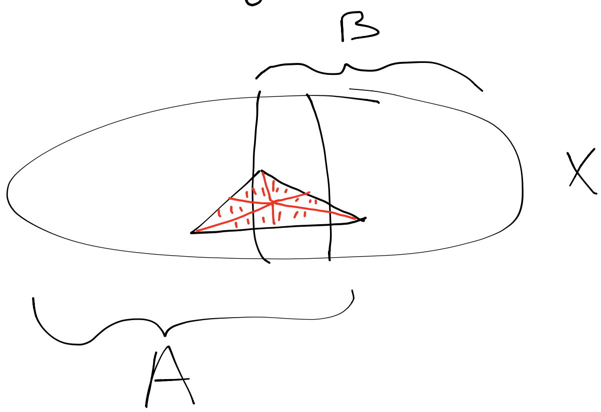
induces an isom. on homology.

Excision follows b/c $(\star) \Rightarrow H_n(B, A\cap B) \cong H_n(A+B, A)$.

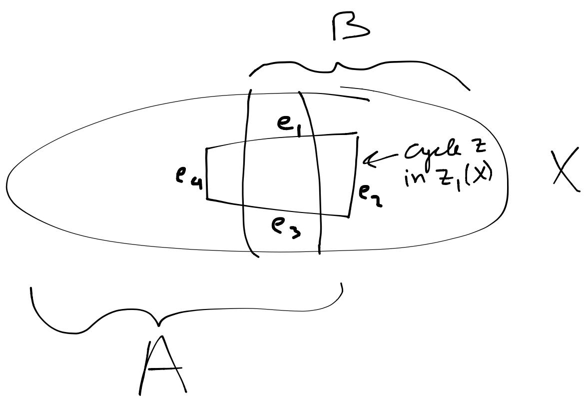
(Note that $(\star)$ is a chain map, since it's induced by the inclusion $C_nB \hookrightarrow C_n(A+B)$.)

Prop 2.1 in Hatcher is a more general version of the Lemma, with $X = A\cup B$ replaced by $X = \bigcup_{i\in I} U_i$ where $\bigcup_{i\in I} \mathrm{int}(U_i) = X$.

The basic idea behind the Lemma is that any chain $\sum x_i \sigma_i$ in $C_n(X)$ can be "cut up", or <u>subdivided</u>, into pieces that lie entirely in A or B:

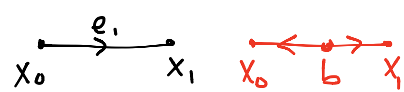


Each red simplex lies entirely on one side of the decomposition.



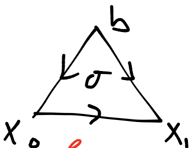
<u>Q</u>: If we cut each edge of z in half (subdivide) why is the resulting cycle in the same homology class?

Compare e_1 & its subdivision:



orientation rule: all edges point away from barycenter b

"Cone off" e_1 to create a triangle $b(e_1)=\sigma$:



σ is the map $\Delta^2 \to X$
$v_0 \mapsto b$
$v_1 \mapsto x_0$
$v_2 \mapsto x_1$

(<u>More formally</u>: $e_1 : \Delta^1 \to X$ & $\sigma : \Delta^2 \to \Delta^1 \xrightarrow{e_1} X$ where $\Delta^2 \to \Delta^1$ is the map
$v_0 \mapsto \frac{1}{2}v_0 + \frac{1}{2}v_1$
$v_1 \mapsto v_0$
$v_2 \mapsto v_1$
& we extend linearly.)

Now $\partial\sigma = [x_0, x_1] - [b, x_1] + [b, x_0]$

So $e_1 = [x_0, x_1] \equiv [b, x_1] - [b, x_0] \pmod{B_1(X)}$

Subdividing each edge of our loop gives a cycle in $C_1(A+B)$ that is "homologous" to $e_1 + e_2 + e_3 + e_4$ (i.e. represents the same class in $H_1 X$).

<u>Pf of Lemma</u>: We'll start by defining a chain map

$S: C_* X \to C_* X$ sending σ to its "subdivision", and we'll construct a chain htpy $T: C_* X \to C_{*+1} X$ with

$$T\partial + \partial T = \mathrm{Id} - S$$

(so T is a htpy b/w S & Id).

S is defined inductively using the <u>cone operators</u>

$$c_b : C_k \Delta^n \to C_{k+1} \Delta^n$$
$$\sigma: \Delta^k \to \Delta^n \longmapsto c_b(\sigma)$$

where $b \in \Delta^n$ is any pt, and $c_b(\sigma): \Delta^{k+1} \to \Delta^n$ is defined by

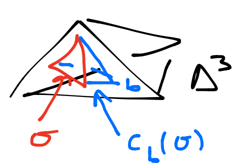
$$c_b(\sigma)\left(\sum_{i=0}^{k+1} \lambda_i e_i\right) = \lambda_0 b + \bar{\lambda}\, \sigma\left(\sum_{i=1}^{k+1} \left(\frac{\lambda_i}{\bar{\lambda}}\right) e_i\right),$$

where $\bar{\lambda} = \sum_{i=1}^{k+1} \lambda_i$.

<u>Note</u>:
For $1 \le i \le k+1$, $0 \le \frac{\lambda_i}{\bar{\lambda}} \le 1$ and $\sum_{i=0}^{k+1} \frac{\lambda_i}{\bar{\lambda}} = \frac{\bar{\lambda}}{\bar{\lambda}} = 1$, so $\sum_{i=1}^{k+1} \left(\frac{\lambda_i}{\bar{\lambda}}\right) e_i \in \Delta^{k+1}$

and moreover, we see that $\lambda_0 b + \bar{\lambda}\, \sigma\left(\sum_{i=1}^{k} \left(\frac{\lambda_i}{\bar{\lambda}}\right) e_i\right) \in \Delta^n$ b/c $\lambda_0 + \bar{\lambda} = 1$.

<u>Picture</u>:

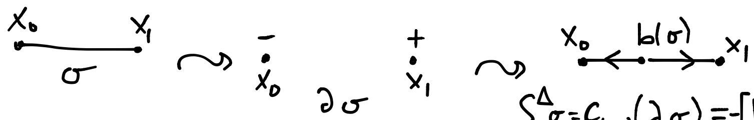


Def: $S^\Delta : C_k\Delta^n \longrightarrow C_k\Delta^n$ is defined inductively:

- For $k=0$, $S^\Delta(\sigma)=\sigma$
- For $k>0$,

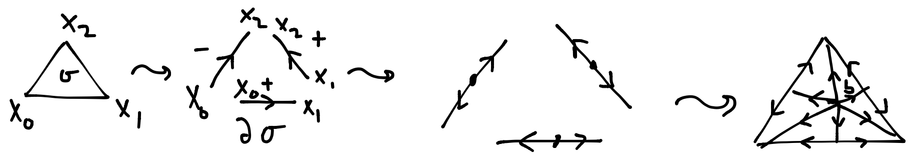
$$S^\Delta(\sigma) = C_{b(\sigma)}\left(S^\Delta(\partial\sigma)\right), \text{ where } b(\sigma)=\sigma\left(\tfrac{1}{n+1},\cdots,\tfrac{1}{n+1}\right).$$

Ex: $k=1$



$S^\Delta\sigma = C_{b(\sigma)}(\partial\sigma) = -[b,x_0]+[b,x_1]$

$k=2$



$S^\Delta(\partial\sigma)$ $\qquad$ $S^\Delta(\sigma) = C_b(S^\Delta\partial\sigma)$

Note: The formulas inductively keep track of the signs on the 6 simplices making up $S^\Delta(\sigma)$, along with the exact parametrizations of these simplices as maps $\Delta^2 \to \Delta^n$.

Defn: For any space X, we set $S : C_nX \to C_nX$ to be $S(\sigma) = \sigma_\#\big(\underbrace{S^\Delta(\mathrm{Id}_{\Delta^n})}_{\text{in } C_n\Delta^n}\big)$.

Here $\sigma_\#$ is the chain map $C_n(\Delta^n) \to C_n(X)$ induced by $\sigma : \Delta^n \to X$.

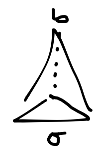
Claim: S is a chain map: $\partial S = S\partial$.

Note: Can see this in pictures above for dim's 0 & 1

The pf requires several steps.

Subclaim 1: $\partial C_b + C_b\partial = \mathrm{Id}$

Subclaim 2: S^Δ is a chain map.

Pf of SC1: **Picture**  $c_b(\sigma)$ $\quad \partial c_b\sigma = \underbrace{\sigma}_{\text{bottom}} \pm \underbrace{c_b(\partial\sigma)}_{\text{sides}}$

Formally: $\partial c_b(\sigma) = \sum_{i=0}^{n+1} (-1)^i c_b(\sigma)\circ\delta^i = c_b(\sigma)\circ\delta^0 - \sum_{i=0}^{n}(-1)^i (c_b\sigma)\circ\delta^{i+1}$

$\bar{t} = \Sigma t_i = 1$

Now $c_b(\sigma)\circ\delta^0 = \sigma$, b/c

$$c_b(\sigma)\circ\delta^0(t_0,\dots,t_n) = c_b(\sigma)(0,t_0,\dots,t_n) = 0\cdot b + \sigma\left(\sum \frac{t_i}{\bar{t}} e_i\right)$$
$$= \sigma(t_0,\dots,t_n).$$

We claim: $\displaystyle\sum_{i=0}^{n}(-1)^i (c_b\sigma)\circ\delta^{i+1} = c_b(\partial\sigma)$

Indeed,
$$c_b(\partial\sigma) = c_b\left(\sum_{i=0}^{n}(-1)^i \sigma\circ\delta^i\right) = \sum_{i=0}^{n}(-1)^i c_b(\sigma\circ\delta^i)$$

so it's enough to check that $c_b(\sigma\circ\delta^i) = c_b(\sigma)\circ\delta^{i+1}$

We have $c_b(\sigma\circ\delta^i)(t_0,\dots,t_n) = t_0 b + \sigma\circ\delta^i(t_1/\bar{t},\dots,t_n/\bar{t})$
$$= t_0 b + \sigma(t_1/\bar{t},\dots,\underset{\substack{\uparrow\\ i^{th}\text{ slot}}}{0},\dots,t_n/\bar{t})$$

and $c_b(\sigma)\circ\delta^{i+1}(t_0,\dots,t_n) = c_b(\sigma)(t_0,\dots,\underset{\substack{\uparrow\\ \text{slot } i+1}}{0},\dots,t_n)$
$$= t_0 b + \sigma\left(\frac{t_1}{\bar{t}},\dots,\overset{\substack{\text{slot } i\\ \downarrow}}{0},\dots,t_n/\bar{t}\right)$$

(in each case, $\bar{t} = t_1 + \dots + t_n$). $\square$

Pf of SC2: By induction on $n = \dim(\sigma)$: $S^\Delta : C_0\Delta^n \to C_0\Delta^n$ is the identity map, so there is nothing to check when $n=0$.

$\underline{n \geq 1}$:

$$\partial S^\Delta\sigma = \partial c_{b(\sigma)}(S^\Delta(\partial\sigma)) \overset{\substack{SC1:\ \partial c_y(\tau) = \tau - c_y(\partial\tau)\\ \downarrow}}{=} S^\Delta(\partial\sigma) - c_{b(\sigma)}(\underbrace{\partial S^\Delta(\partial\sigma)}_{\substack{\text{Ind. Hypoth.}\ \to\ \parallel\\ S^\Delta(\partial\partial\sigma)=0}})$$
$$= S^\Delta(\partial\sigma). \ \square$$

Finally we prove the claim $(\partial S = S\partial)$:

$$\partial S(\sigma) = \partial\, \sigma_\# S^\Delta(\mathrm{Id}_{\Delta^n}) \overset{\substack{\sigma_\# \text{ is a chain map}\\ \downarrow}}{=} \sigma_\#\left(\partial S^\Delta(\mathrm{Id}_{\Delta^n})\right)$$

$$\overset{\substack{S\mathcal{C}2\\ \downarrow}}{=} \sigma_\#\left(S^\Delta(\partial\, \mathrm{Id}_{\Delta^n})\right) = \sigma_\#\left(S^\Delta\left(\sum_{i=0}^n (-1)^i \delta^i\right)\right)$$

$$= \sum_{i=0}^n (-1)^i \sigma_\# S^\Delta(\delta^i)$$

-he other hand,

$$S\partial(\sigma) = S(\partial\sigma) = S\left(\sum_{i=0}^n (-1)^i \sigma\circ\delta^i\right) = \sum_{i=0}^n (-1)^i S(\sigma\circ\delta^i)$$

$$= \sum_{i=0}^n (-1)^i (\sigma\circ\delta^i)_\# \left(S^\Delta(\mathrm{Id}_{\Delta^{n-1}})\right)$$

$$= \sum_{i=0}^n (-1)^i \sigma_\# \circ (\delta^i)_\# \left(S^\Delta(\mathrm{Id}_{\Delta^{n-1}})\right)$$

So it suffices to check that

$$(\delta^i)_\# S^\Delta(\mathrm{Id}_{\Delta^{n-1}}) = S^\Delta(\delta^i)$$

<u>Fact</u>: $\tau_\# S^\Delta(\eta) = S^\Delta(\tau\circ\eta)$ for all $\eta : \Delta^k \to \Delta^{n-1}$ and all <u>linear</u> maps $\tau : \Delta^{n-1} \to \Delta^n$ (that is, $\tau(\Sigma \lambda_i e_i) = \Sigma \lambda_i \tau(e_i)$).

To prove this, we need a similar observation about the cone operator C_b:

<u>Exercise</u>: $\tau_\# C_b(v) = C_{\tau(b)}(\tau_\# v)$ so long as τ is linear.

(here $v \in C_* \Delta^n$).

Pf of Fact: For $\dim(\eta)=0$, this is immediate from the def'ns.

We proceed by induction. Assuming the statement in dim'n $n-1$, if $\dim\eta = n$ we have

$$\tau_\# S^\Delta(\eta) = \tau_\# c_{b(\eta)}\left(S^\Delta(\partial\eta)\right) \underset{\substack{\uparrow \\ \text{by the Exercise}}}{=} c_{\tau(b(\eta))}\left(\tau_\# S^\Delta(\partial\eta)\right)$$

$S^\Delta, \tau_\#$ are linear maps

$$\overset{\downarrow}{=} c_{\tau(b(\eta))}\left(\sum_{i=0}^{n}(-1)^i\,\tau_\# S^\Delta(\eta\circ\delta^i)\right)$$

Induction Hypothesis

$$\overset{\downarrow}{=} c_{\tau(b(\eta))}\left(\sum_{i=0}^{n}(-1)^i S^\Delta(\tau\circ\eta\circ\delta^i)\right)$$

By linearity of τ,

$$\tau(b(\eta)) = \tau\left(\sum_{i=0}^{n}\tfrac{1}{n+1}\eta(e_i)\right) = \sum_{i=0}^{n}\tfrac{1}{n+1}\tau\circ\eta(e_i) = b(\tau\circ\eta)$$

So the above is just

$$c_{b(\tau\circ\eta)}\left(S^\Delta(\partial(\tau\circ\eta))\right)$$ as desired. $\square$

This completes the proof that S is a chain map.

Next we need:

Claim: There is a chain htpy $T: C_n X \to C_{n+1}X$ satisfying $\partial T + T\partial = \mathrm{Id} - S$.

Hatcher constructs T^Δ on $C_*\Delta^n$ first, by induction and then extends via $T(\sigma) = \sigma_\# T^\Delta(\mathrm{Id}_{\Delta^n})$, just like the construction of S. The inductive formula for T^Δ is

$$T^\Delta(\sigma) = c_{b(\sigma)}(\sigma) - c_{b(\sigma)}\left(T^\Delta(\partial\sigma)\right)$$

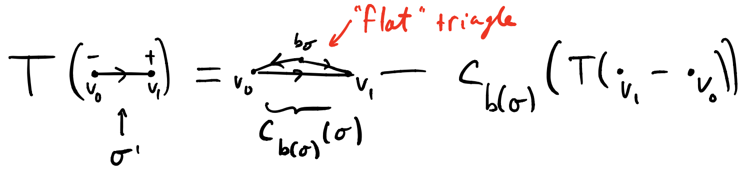
$T^\Delta: C_0\Delta^n \to C_1\Delta^n$
$x \mapsto \underbrace{c_x(x)}_{\text{const. map } \Delta^1 \to \{x\}}$

The details in the proof of this Claim are similar to what we did to understand ∂S, and are basically formal. (Details below.)

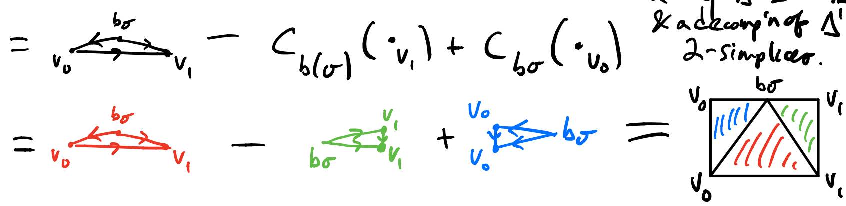
Pictures: $T(\cdot)$ = constant 1-simplex w/ same image

$\uparrow$ 0-simplex

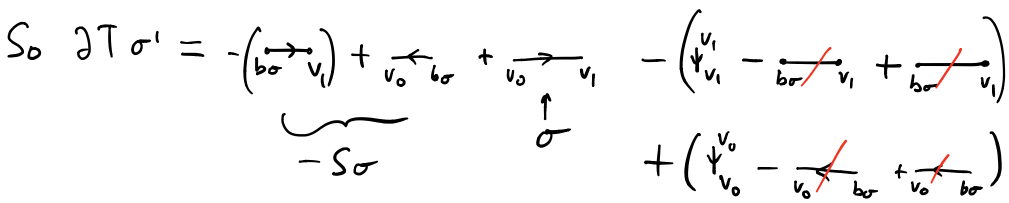
Note: $\partial T(\cdot) + T\partial \cdot = 0 = \cdot - S(\cdot)$



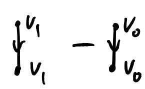
This picture describes a map $\Delta^1 \times I \to \Delta^1 \xrightarrow{\sigma} X$ & a decomp'n of $\Delta^1 \times I$ into 3 2-simplices.

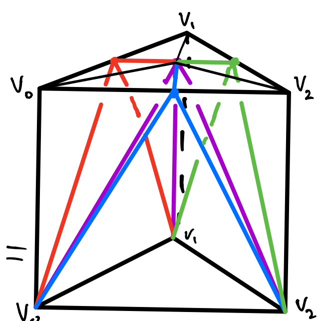


So $\partial T\sigma' =$

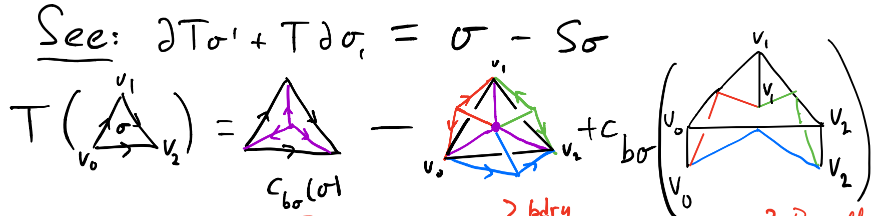


and $T(\partial\sigma') = T(v_1 - v_0) =$





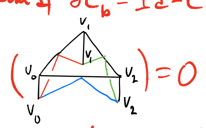
See: $\partial T\sigma' + T\partial\sigma_1 = \sigma - S\sigma$



bdry splits into 3 terms in this column

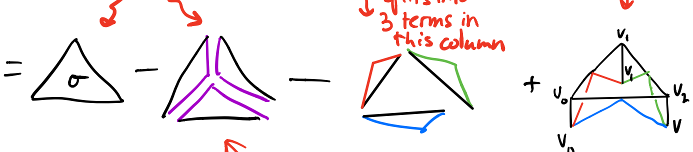
Recall (Subclaim 1): $\partial C_b = Id - C_b\partial$

and $\partial(\ldots) = 0$



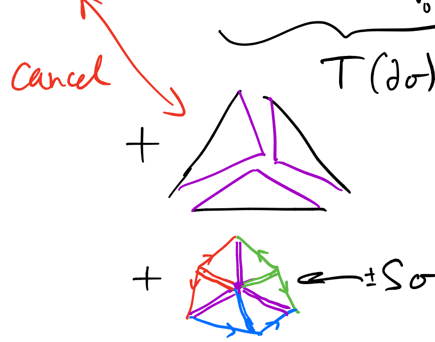
b/c of the repeated vertices

$\partial T(\sigma) =$



$T(\partial\sigma)$

Cancel



<u>Proof of $\partial T + T\partial = Id - S$:</u> First we check that $\partial T^\Delta + T^\Delta \partial = Id - S^\Delta$, and then the general result follows formally.

To prove $\partial T^\Delta + T^\Delta\partial = Id - S^\Delta$ on $C_n\Delta^m$, we proceed by induction on $\underline{n}$. When $n=0$, we have $\partial T^\Delta \sigma^0 + T^\Delta \partial \sigma^0 = \partial(\underbrace{c_{Im(\sigma^0)}}_{\text{constant 1-simplex}}) + 0 = 0$

and $Id(\sigma^0) - S^\Delta(\sigma^0) = \sigma^0 - \sigma^0 = 0$ as well.

Now assume the result for $(n-1)$-simplices and compute $\partial T^\Delta \sigma^n$ for $\sigma^n \in C_n\Delta^m$:

$$\partial T^\Delta \sigma = \partial c_{b\sigma}(\sigma) - \partial c_{b\sigma}(T^\Delta \partial \sigma) \overset{\partial c_b \tau = \tau - c_b(\partial \tau) \text{ by Subclaim 1}}{=} (\sigma - c_{b\sigma}(\partial\sigma)) - (T^\Delta \partial\sigma - c_{b\sigma}(\partial T^\Delta \partial \sigma))$$

We want to show this equals $\sigma - S^\Delta\sigma - T^\Delta\partial\sigma$, so it suffices to show

$$(\star) \qquad c_{b\sigma}(\partial\sigma) - c_{b\sigma}(\partial T^\Delta \partial \sigma) = S^\Delta \sigma.$$

So we compute $\partial T^\Delta(\partial\sigma)$:

$$\partial T^\Delta(\partial\sigma) = \partial T^\Delta\left(\sum (-1)^i \sigma \circ \delta^i\right) = \sum (-1)^i \partial\left(T^\Delta(\underbrace{\sigma\circ\delta^i}_{(n-1)\text{-simplex}})\right)$$

$$\overset{\text{By induction}}{=} \sum (-1)^i \left(\sigma\circ\delta^i - S^\Delta(\sigma\circ\delta^i) - T^\Delta\partial(\sigma\circ\delta^i)\right)$$

$$= \partial\sigma - S^\Delta(\partial\sigma) - T^\Delta(\underbrace{\partial(\partial\sigma)}_{0})$$

$$\overset{S^\Delta\partial = \partial S^\Delta}{=} \partial\sigma - \partial S^\Delta \sigma$$

So $(\star)$ becomes

$$\cancel{c_{b\sigma}(\partial\sigma)} - c_{b\sigma}(\cancel{\partial\sigma} - \partial S^\Delta \sigma) = S^\Delta \sigma,$$

and this is the <u>defn</u> of $S^\Delta\sigma$.

For $\sigma \in C_n X$, we have:
$$\begin{aligned}\partial T\sigma + T\partial\sigma &= \partial \sigma_\# T^\Delta(Id_{\Delta^n}) + T(\partial\sigma)\\ &= \sigma_\# \partial T^\Delta(Id_{\Delta^n}) + T(\partial\sigma)\\ &= \sigma_\#\left(Id_{\Delta^n} - S^\Delta Id_{\Delta^n} - T^\Delta(\partial Id_{\Delta^n})\right) + T(\partial\sigma)\\ &= \sigma - S\sigma - \sigma_\# T^\Delta(\partial Id_{\Delta^n}) + T(\partial\sigma)\end{aligned}$$

To complete the proof, we need to show that

$$T(\partial\sigma) = \sigma_\# T^\Delta(\partial Id_{\Delta^n}).$$

We have:

$$T(\partial\sigma) = \sum(-1)^i T(\sigma\circ\delta^i) = \sum(-1)^i (\sigma\circ\delta^i)_\# T^\Delta(Id_{\Delta^{n-1}})$$
$$= \sigma_\#\left(\sum(-1)^i \delta^i_\# T^\Delta(Id_{\Delta^{n-1}})\right)$$

and

$$\sigma_\#\left(T^\Delta(\partial Id_{\Delta^n})\right) = \sigma_\#\left(T^\Delta\left(\sum(-1)^i\delta^i\right)\right)$$
$$= \sigma_\#\left(\sum(-1)^i T^\Delta(\delta^i)\right)$$

So it suffices to show that if $f:\Delta^m\to\Delta^\ell$ is <u>linear</u>, & $\omega\in C_k(\Delta^m)$, then

$$f_\#(T^\Delta\omega) = T^\Delta(f\circ\omega). \quad [\text{set } f=\delta^i,\ \omega=Id_{\Delta^{n-1}}]$$

We prove this by induction on k, using the above exercise.

For $k=0$, both sides are the constant 1-simplex at the image of $f\omega$.

Now assume the result for $k-1$.

$$f_\# T^\Delta(\omega) = f_\#\left(C_{b\omega}(\omega) - C_{b\omega}(T^\Delta(\partial\omega))\right)$$

By the Exercise

$$\overset{\downarrow}{=} C_{f(b\omega)}(f\omega) - C_{f(b\omega)}(f_\# T^\Delta\partial\omega)$$

$$T^\Delta(f\circ\omega) = C_{b(f\omega)}(f\omega) - C_{bf\omega}(T^\Delta(\partial f\omega))$$

Since f is linear, $f(b\omega)=b(f\omega)$. So it remains only to show

$$f_\# T^\Delta(\partial\omega) = T^\Delta(\partial f\omega).$$

<u>We have:</u> $f_\# T^\Delta(\partial\omega) = \sum(-1)^i f_\# T^\Delta(\omega\circ\delta^i) = \sum(-1)^i T^\Delta(f\circ\omega\circ\delta^i)$

$$= T^\Delta(\partial(f\circ\omega)) \ \square$$

Now we can complete the proof of excision.

We need to show that $C_n(A+B, A) \longrightarrow C_n(X, A)$ induces an isom. on homology. First we prove surjectivity.

<u>Lemma</u>: (Hatcher, p. 120) Given $z = \sum \lambda_i \sigma_i \in C_n(X)$ w/ $\partial z \in C_{n-1}(A)$, $S^m(z) \in C_n(A+B)$ for some $m = m(z) \geq 0$.

ε is the Lebesgue # of the cover $\{\sigma_i^{-1}(A), \sigma_i^{-1}(B)\}_i$

<u>Pf</u>: (Sketch) For each i, $\sigma_i^{-1}(A), \sigma_i^{-1}(B) \subseteq \Delta^n$, and hence $\exists\ \varepsilon > 0$ s.t. each ball of rad. ε (in Δ^n) maps into either A or B under all σ_i. Now one checks inductively that the diameter of simplices in the m^{th} subdivision of Δ^n is $\leq (n/n+1)^m$, which is $< \varepsilon$ for $m \gg 0$. $\square$

Now it suffices to show that $\partial(S^m(z)) \in C_{n-1}(A)$ (so that $S^m(z)$ is a cycle in $C_*(A+B, A)$) and that $z - S^m(z)$ lies in $B_n(X) + C_n(A)$ (so that z & $S^m z$ give the same homology class in $H_*(X, A) = \dfrac{\{z \in C_n X \mid \partial z \in C_n A\}}{B_n X + C_n A}$.

We have $\partial S^m(z) = S^m(\partial z)$ (b/c $\partial S = S \partial$) and $\partial z \in C_{n-1} A$. From the defn of S we see that $S(C_* A) \subseteq C_* A$, so $\partial S^m(z) \in C_{n-1} A$ as desired.

Next, $z - S^m(z)$ is a bdry in $C_*(A+B, A)$ b/c can write

$$z - S^m z = z - Sz + (Sz - S^2 z) + \cdots + (S^{m-1} z - S^m z)$$
$$= \sum_{i=0}^{m-1} S^i(z - Sz)$$

and $z - Sz = \partial T z - T \partial z \in B_n X + C_n A$ b/c $\partial z \in C_{n-1} A$, and S^i maps $B_n X$ to $B_n X$ (since S is a chain map) and maps $C_n A$ to $C_n A$.

Finally, we show $C_n(A+B,A) \to C_n(X,A)$ is injective on homology. Say $z \in C_n(A+B,A)$ and z represents $0 \in H_n(X,A)$. Then $\partial z \in C_n A$ and

$$z \in B_n X + C_n A,$$

so we can write $z = \partial w + \alpha$ with $w \in C_{n+1}(X)$, $\alpha \in C_n(A)$. We want to show that $z \in B_n(A+B) + C_n(A)$, and it suffices to show that $\partial w = z - \alpha \in B_n(A+B)$. Again, consider the telescoping sum

$$w - S^m w = (w - Sw) + (Sw - S^2 w) + \cdots + (S^{m-1}w + S^m w)$$

where m is taken large enough that $S^m(w) \in C_n(A+B)$.

Now

$$\partial w = \underbrace{\partial S^m w}_{\text{in } B_n(A+B)} + \partial\left[\sum_{i=0}^{m-1} S^i(w - Sw)\right] = \partial S^m w + \sum_{i=0}^{m-1} S^i(\partial(w - Sw))$$

so it will suffice to show that $\partial(w - Sw) \in B_n(A+B)$.

We have

$$\partial(w - Sw) = \partial(\partial T w + T\partial w) = \partial T \partial w = \partial T(z - \alpha)$$

Since $z \in C_n(A+B)$ & $\alpha \in C_n A$, $z - \alpha \in C_n(A+B)$ and hence $T(z-\alpha) \in C_n A+B$ as well. So

$$\partial(w - Sw) = \partial T(z - \alpha) \in B_n(A+B),$$

as desired. This completes the proof of excision. □

The Mayer-Vietoris Sequence:

To derive the M-V seq. for $X = A\cup B$ (w/ $\text{int}(A)\cup\text{int}(B)=X$) we consider

$$\begin{tikzcd} & A \arrow[dr, hook, "i^A"] & \\ A\cap B \arrow[ur, hook, "i_A"] \arrow[dr, hook, "i_B"'] \arrow[rr, hook, "i"] & & X \\ & B \arrow[ur, hook, "i^B"'] & \end{tikzcd}$$

$$C_n(A\cap B) \xrightarrow{(i_{A\#},\, -i_{B\#})} C_n(A)\oplus C_n(B) \xrightarrow{i^A_\# + i^B_\#} C_n(A+B)$$

The 2^{nd} map is onto by def'n of $C_n(A+B)$, and the first is clearly injective. In fact, the seq. is exact b/c

$$i^A_\# + i^B_\#\,(i_{A\#}\sigma, -i_{B\#}\sigma) = i_\#\sigma - i_\#\sigma = 0.$$

Now the LES in H_* has the form

$$\cdots \to H_n(A\cap B) \longrightarrow H_n(A)\oplus H_n B \longrightarrow H_n(A+B) \xrightarrow{\partial} H_{n-1}(A\cap B) \to \cdots$$

<u>Claim</u>: The inclusion $C_*(A+B) \overset{i_\#}{\hookrightarrow} C_*(X)$ induces isom's

$$H_n(A+B) \cong H_n(X)$$

for all $n\geq 0$.

<u>Pf</u>: We have a comm. diagram of SES

$$\begin{tikzcd} C_*A \arrow[r] \arrow[d, "="] & C_*(A+B) \arrow[r] \arrow[d, "i_\#"] & C_*(A+B, A) \arrow[d] \\ C_*A \arrow[r] & C_*(X) \arrow[r] & C_*(X,A) \end{tikzcd}$$

and the right-most vertical map induces an isom. on H_* by Prop 2.21. The claim now follows by applying the 5-lemma to the associated diagram of LES's in homology. □

In fact, the MV seq. can also be formulated for reduced homology: just use the augmented chain cplxs

$$\begin{array}{ccc} & & C_n Z \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ \sum n_i x_i & & C_0 Z \\ \big\downarrow & & \downarrow \varepsilon \\ \sum n_i & & \mathbb{Z} \end{array}$$

for $Z = A \cap B, A, B$, (the homology of this augmented cplx is $\tilde{H}_*(Z)$) and note that $C_0 A{+}B = C_0 X$ so we can augment $C_*(A{+}B)$ in the same manner.

Note that we still have a SES of augmented cplxs if the maps in deg -1 are $\underset{1 \mapsto (1,-1)}{\mathbb{Z} \longrightarrow} \mathbb{Z} \oplus \mathbb{Z} \underset{(a,b) \mapsto a+b}{\longrightarrow \mathbb{Z}}$.