THE SPECTRAL SEQUENCE OF A TOWER OF COFIBRATIONS AND THE LAWSON SPECTRAL SEQUENCE

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Abstract. In this note, we discuss the spectral sequence associated to a sequence of cofibrations of spectra, and a resulting spectral sequence due to Lawson involving spaces of irreducible representations. We also give a brief description of an example of this spectral sequence for surface groups.

1. The spectral sequence of a tower of cofibrations

Given a sequence of spectra

$$* = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots,$$

we set $C_i = \text{hocofib}(X_i \to X_{i+1})$ and $X = \text{hocolim} X_i$.

Proposition 1.1. *There is a spectral sequence*

$$E^1_{p,q} = \pi_{p+q} C_p \implies \pi_{p+q} X,$$

in which the differential on the r*th page has the form*

$$d^r_{p,q}\colon E^1_{p,q} \longrightarrow E^1_{p-r,q+r-1}.$$

We now sketch the construction of this sequence. There is then an exact couple

$$\begin{array}{ccc} A^1 & \xrightarrow{\;i_*\;} & A^1 \\ {\scriptstyle \partial}\nwarrow & & \swarrow{\scriptstyle j_*} \\ & E^1 & \end{array}$$

in which

$$A^1 = \bigoplus_{n,p} \pi_n X_p$$

and

$$E^1 = \bigoplus_{n,p} \pi_n C_p.$$

The maps i and j are the direct sums of the maps on homotopy induced by $i_p\colon X_p \to X_{p+1}$ and $j_p\colon X_p \to C_p$, respectively, and ∂ is the direct sum of the boundary maps $\partial\colon \pi_n C_p \to \pi_{n-1} X_{p-1}$. Abusing notation, we will denote the composition of r adjacent maps $(i_p)_*$ by i^r_* (and i^0_* will be interpreted as the identity map). This exact couple gives rise to a spectral sequence, whose rth page we denote by E^r.

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Lemma 1.2. *For $r \geqslant 1$, there are isomorphisms $E^r \cong \bigoplus_{n,p} \widetilde{E}^r_{n,p}$, where*

$$\widetilde{E}^r_{n,p} = \frac{\{a \in \pi_n C_p \mid \partial a \in \mathrm{Im}(i_*^{r-1}\colon \pi_{n-1}X_{p-r} \to \pi_{n-1}X_{p-1})\}}{\{a \in \pi_n C_p \mid a = j_* b,\, b \in \ker(i_*^{r-1}\colon \pi_n X_p \to X_{p+r-1})\}},$$

and the differential d_r is corresponds under this isomorphism to the direct sum of the maps

$$d_r^{n,p}\colon \widetilde{E}^r_{n,p} \longrightarrow \widetilde{E}^r_{n-1,p-r}$$

given by

$$[a] \mapsto [j_* b],$$

where $b \in \pi_{n-1}X_{p-r}$ satisfies $i_*^r(b) = \partial a$. (Note that $j_* b$ represents a class in $\widetilde{E}^r_{n,p}$, since $\partial(j_* b) = 0$ by exactness.)

Letting

$$Z^r_{n,p} = \{a \in \pi_n C_p \mid \partial a \in \mathrm{Im}(i_*^r\colon \pi_{n-1}X_{p-r} \to \pi_{n-1}X_p)\}$$

for $r \geqslant 1$ (note that $Z^1_{n,p} = \pi_n C_p$) and

$$B^r_{n,p} = \{a \in \pi_n C_p \mid a = (j_p)_* b,\, b \in \ker(i_*^{r-1}\colon \pi_n X_p \to \pi_n X_{p+r-1})\}$$

for $r \geqslant 1$ we have

$$0 = B^1_{n,p} \subset B^2_{n,p} \subset \cdots \subset B^r_{n,p} \subset \cdots \subset Z^r_{n,p} \subset Z^2_{n,p} \subset Z^1_{n,p} = \pi_n C_p$$

and by definition,

$$\widetilde{E}^r_{n,p} = Z^r_{n,p}/B^r_{n,p}.$$

We define

$$Z^\infty_{n,p} = \bigcap_{r \geqslant 1} Z^r_{n,p}, \quad B^\infty_{n,p} = \bigcup_{r \geqslant 1} B^r_{n,p}, \quad \text{and } \widetilde{E}^\infty_{n,p} = Z^\infty_{n,p}/B^\infty_{n,p}.$$

Note that

$$Z^\infty_{n,p} = \ker(\partial\colon \pi_n C_p \to \pi_{n-1}X_{p-1})$$

and

$$B^\infty_{n,p} = \{a \in \pi_n C_p \mid a = j_* b,\, b \in \ker(i_*^\infty\colon \pi_n X_p \to \pi_n X)\}.$$

Lemma 1.3. *Letting i_*^∞ denote the map(s) $\pi_* X_p \to \pi_* X$, we have a filtration*

$$0 \subset i_*^\infty \pi_n X_1 \subset i_*^\infty \pi_n X_2 \subset \cdots.$$

The associated graded groups of this filtration satisfy

$$\frac{i_*^\infty \pi_n X_p}{i_*^\infty \pi_n X_{p-1}} \cong \widetilde{E}^\infty_{n,p}.$$

In light of the lemma, we may write

$$\widetilde{E}^1_{n,p} \implies \pi_n X,$$

or in Serre indexing, with $E^1_{p,q} = \widetilde{E}^1_{p+q,p}$,

$$E^1_{p,q} = \pi_{p+q} C_p \implies \pi_{p+q} X,$$

as claimed.

2. The Lawson Spectral Sequence

Lawson examined a particular case of the spectral sequence from Section 1 in [1, 2]. Given a discrete group Γ, the unitary representation spaces

$$\mathrm{Hom}(\Gamma, U(n))/U(n)$$

combine to form an abelian topological monoid under block sum. Specifically, we define

$$\mathrm{Rep}(\Gamma) = \coprod_{n=0}^{\infty} \mathrm{Hom}(\Gamma, U(n))/U(n).$$

(When $n = 0$, we define $\mathrm{Hom}(\Gamma, U(n))/U(n)$ to be a point, which serves as the identity element of this monoid.

For each $k \geqslant 0$, we may consider the submonoid $\mathrm{Rep}_k(\Gamma) \subset \mathrm{Rep}(\Gamma)$ generated by

$$\coprod_{n=0}^{k} \mathrm{Hom}(\Gamma, U(n))/U(n).$$

In other words, $\mathrm{Rep}_k(\Gamma)$ consists of those representations whose irreducible summands all have dimension at most k.

For any topological abelian monoid A, the bar construction BA is again a topological abelian monoid, and by iterating this construction one obtains an Ω–spectrum of the form

$$\Omega BA,\ BA,\ B(BA),\ \cdots.$$

The spectrum associated to $\mathrm{Rep}_k(\Gamma)$ in this manner is denoted $R_k^{\mathrm{def}}(\Gamma)$, so we have

$$\Omega^\infty R_k^{\mathrm{def}}(\Gamma) = \Omega B \mathrm{Rep}_k(\Gamma).$$

Similarly, the spectrum associated to $\mathrm{Rep}(\Gamma)$ is denoted $R^{\mathrm{def}}(\Gamma)$, so we have

$$\Omega^\infty R^{\mathrm{def}}(\Gamma) = \Omega B \mathrm{Rep}(\Gamma).$$

Lawson proves that the inclusions $\mathrm{Rep}_k\Gamma \hookrightarrow \mathrm{Rep}_{k+1}\Gamma$ induce cofibrations

$$R_k^{\mathrm{def}}(\Gamma) \hookrightarrow R_{k+1}^{\mathrm{def}}(\Gamma)$$

and it follows that

$$R^{\mathrm{def}}(\Gamma) = \operatorname*{telescope}_k R_k^{\mathrm{def}}(\Gamma).$$

(Lawson's proof relies on results from Park–Suh [3] about actions of algebraic groups, but this can also be proven using the semi-algebraic methods discussed in Ramras [4].)

Lawson shows that there are homotopy cofiber sequences of spectra

$$R_{k-1}^{\mathrm{def}}(\Gamma) \longrightarrow R_k^{\mathrm{def}}(\Gamma) \longrightarrow H\mathbb{Z} \wedge \overline{\mathrm{Irr}}_k^+ \Gamma,$$

where $\overline{\mathrm{Irr}}_{k+1}^+\Gamma$ is the one-point compactification[1] of the moduli space of irreducible representations of Γ and $H\mathbb{Z}$ is the Eilenberg–Maclane spectrum for the integers. Since $H\mathbb{Z}$ represents integral homology theory, we have

$$\pi_n H\mathbb{Z} \wedge \overline{\mathrm{Irr}}_k^+ \Gamma = \widetilde{H}_n \overline{\mathrm{Irr}}_k^+.$$

[1] Alternatively, $\overline{\mathrm{Irr}}_k^+\Gamma$ may be described as the quotient space

$$\left(\mathrm{Hom}(\Gamma, U(k))/U(k)\right) / \left(\mathrm{Sum}(\Gamma, U(k))/U(k)\right),$$

where $\mathrm{Sum}(\Gamma, U(k)) \subset \mathrm{Hom}(\Gamma, U(k))$ denotes the subspace of reducible representations.

The spectral sequence associated to the tower

$$* \longrightarrow R_1^{\mathrm{def}}\Gamma \longrightarrow R_2^{\mathrm{def}}\Gamma \longrightarrow R_3^{\mathrm{def}}\Gamma \longrightarrow \cdots$$

now has the form

$$E^1_{p,q} = \pi_{p+q} H\mathbb{Z} \wedge \overline{\mathrm{Irr}}_p^+ \Gamma = H_{p+q} \overline{\mathrm{Irr}}_p^+ \implies \pi_{p+q} R^{\mathrm{def}}(\Gamma),$$

with differentials

$$d_r^{n,p} \colon E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$

3. Example

We consider the case of a closed, orientable surface M^2 of genus 2. In this case, the homotopy groups of $R^{\mathrm{def}}(\pi_1 M^2)$ were computed in Ramras [5] using Yang–Mills theory. In fact, these groups were shown to be isomorphic to the integral (co)homology groups of M^2, so we have

$$\pi_* R^{\mathrm{def}}(\pi_1(M_2)) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } * = 1 \\ 0 & \text{else.} \end{cases}$$

On the other hand, when $n = 1$ we have

$$\overline{\mathrm{Irr}}_1^+(\pi_1 M^2) \cong (S^1)^4_+,$$

where $+$ denotes a disjoint basepoint (note here that we are one-point compactifying a space that is already compact). The homology groups of this torus appear on the line $p = 1$ in the E^1 page of the Lawson spectral sequence, and our knowledge of the E^∞ page shows that there must be classes in higher rank to kill off the excess. Some of these classes are simply induced by pulling back representations along the pinch map $M^2 \longrightarrow (S^1)^2 \vee (S^1)^2$. An analysis of the Lawson spectral sequence for $\pi_1((S^1)^2 \vee (S^1)^2) = \mathbb{Z}^2 * \mathbb{Z}^2$, along with the (split) maps of spectral sequences arising from the (split) inclusions $\mathbb{Z}^2 \hookrightarrow \mathbb{Z}^2 * \mathbb{Z}^2 \hookleftarrow \mathbb{Z}^2$ allows one to conclude that there is a class in $H_2(\overline{\mathrm{Irr}}_1^+(\pi_1 M^2))$ that is *not* killed by classes induced from $(S^1)^2 \vee (S^1)^2$. In conclusion, we find the following.

Proposition 3.1. *There exists a non-torsion class in $H_3(\overline{\mathrm{Irr}}_k^+(\pi_1 M^2))$, for some $k \geqslant 2$, that is not in the image of the map*

$$H_3(\overline{\mathrm{Irr}}_k^+(\mathbb{Z}^2 * \mathbb{Z}^2)) \longrightarrow H_3(\overline{\mathrm{Irr}}_k^+(\pi_1 M^2)).$$

Preliminary computations suggest that similar statements can be made for higher genus surfaces. It would of course be interesting to get a better handle on the differentials in this spectral sequence, so as to (hopefully) identify the dimension of the class described in the Proposition.

References

[1] Tyler Lawson. The product formula in unitary deformation K-theory. *K-Theory*, 37(4):395–422, 2006.

[2] Tyler Lawson. The Bott cofiber sequence in deformation K-theory and simultaneous similarity in $U(n)$. *Math. Proc. Cambridge Philos. Soc.*, 146(2):379–393, 2009.

[3] Dae Heui Park and Dong Youp Suh. Linear embeddings of semialgebraic G-spaces. *Math. Z.*, 242(4):725–742, 2002.

[4] Daniel A. Ramras. Periodicity in the stable representation theory of crystallographic groups. To appear in Forum Math., 2011.

[5] Daniel A. Ramras. The stable moduli space of flat connections over a surface. *Trans. Amer. Math. Soc.*, 363(2):1061–1100, 2011.

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