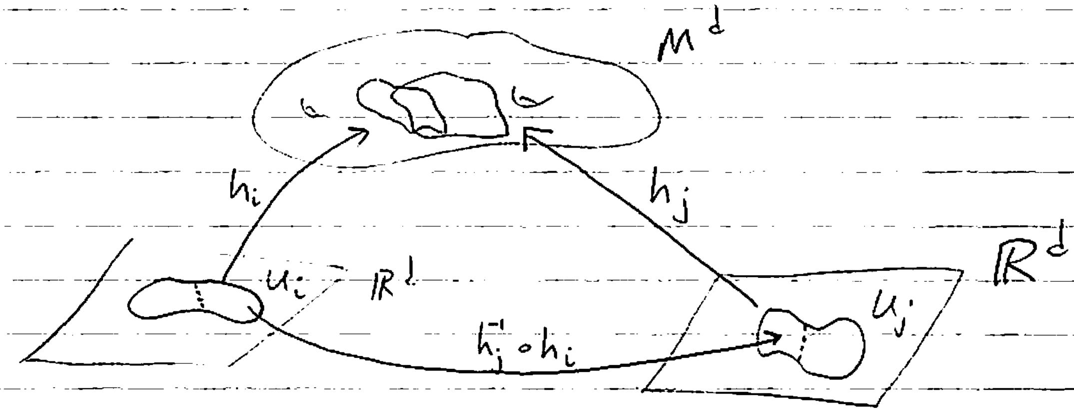
Smooth Manifolds and Their Tangent Bundles

Tangent bundles to smooth manifolds are one of the oldest and most important examples of vector bundles.

Def'n: A smooth structure on a d-dim'l mfld M^d is a collection of charts $h_i : U_i \to M^d$ (U_i open in $\mathbb{R}^d$) whose images cover M^d and whose transition functions $h_j^{-1} \circ h_i$ are smooth.



- Note that $h_j^{-1} \circ h_i$ is defined on $h_i^{-1}\big(h_j(U_j) \cap h_i(U_i)\big)$.

- We call a map $h : U \to V$ ($U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ open sets) smooth if all partial derivatives of all orders exist and are continuous.

Tangent bundles to smooth manifolds can be described abstractly (see Spivak, Diff. Geom. Vol. I), but it is much easier to define tangent bundles to submanifolds of $\mathbb{R}^n$, because we already know how to define tangent vectors in $\mathbb{R}^n$:

Following Milnor + Stasheff, we have:

__Def'n:__ A subset $M^d \subseteq \mathbb{R}^N$ is a __(sub)-manifold__ of __dimension d__ if at each point $x \in M^d$ there exists an open set $U \subseteq \mathbb{R}^d$ and a smooth map $h : U \longrightarrow M^d$ (i.e. $U \xrightarrow{h} M^d \hookrightarrow \mathbb{R}^N$ is smooth)

"open subset"

such that

1) h is a homeomorphism onto its image, and $h(U) \overset{\circ}{\subseteq} M$
2) the vectors $\frac{\partial h}{\partial u_j} = \begin{pmatrix} \frac{\partial h_1}{\partial u_j} \\ \vdots \\ \frac{\partial h_N}{\partial u_j} \end{pmatrix} \in \mathbb{R}^N$ $(j = 1, \cdots, d)$ are linearly independent
   (i.e. the Jacobian $\left(\frac{\partial h_i}{\partial u_j}(a)\right)$ has maximum rank).

[We call h a __local parametrization__.]

__Remarks__
- Condition 1) implies that M^d is a topological mfld.
- Condition 2) implies that M^d is a __smooth__ mfld: Using the Implicit Fcn Theorem one can check that the transition fcns $g^{-1} \circ h$ are automatically smooth (
  (Lemma 1.1 in Milnor - Stasheff).
- The notation $\frac{\partial h_i}{\partial u_j}$ means we write $h(a) = (h_1(a), \cdots, h_N(a))$ $(a \in \mathbb{R}^d)$ and then $$\frac{\partial h_i}{\partial u_j}(\vec{a}) = \lim_{t \to 0} \frac{h_i(a_1, \cdots, a_j + t, \cdots, a_d) - h_i(\vec{a})}{t}.$$
  $(\vec{a} = (a_1, \cdots, a_d))$

We now want to define __tangent vectors__ to points $x \in M$. These arise as __velocity vectors__ to smooth paths through x.

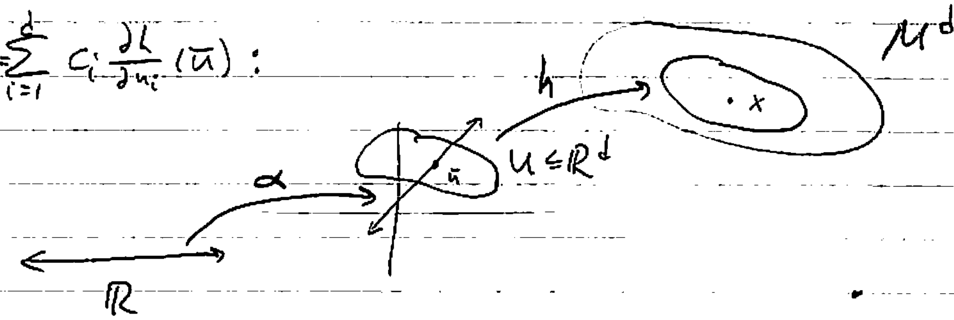
__Def'n:__ If $\alpha : \underset{\subseteq \mathbb{R}}{[a,b]} \to M^d \subseteq \mathbb{R}^N$ is smooth and $\alpha(t_0) = x$, then the velocity vector of α at t_0 is $\alpha'(t_0) = \left(\frac{d\alpha_1}{dt}(t_0), \cdots, \frac{d\alpha_N}{dt}(t_0)\right)$.

<u>Def'n</u>: The <u>tangent space</u> $T_x M^d$ to $M^d \subseteq \mathbb{R}^N$ at a point $x \in M^d$ is $\{\vec{v} \in \mathbb{R}^N \mid \vec{v} = \alpha'(t)$ for some smooth path $\alpha : \mathbb{R} \to M^d$ w/ $\alpha(t) = x\}$.

<u>Lemma 1.2</u>: Say $h : U \to M^d$ is a local parametrization of M^d, with $x = h(\bar{u})$. Then

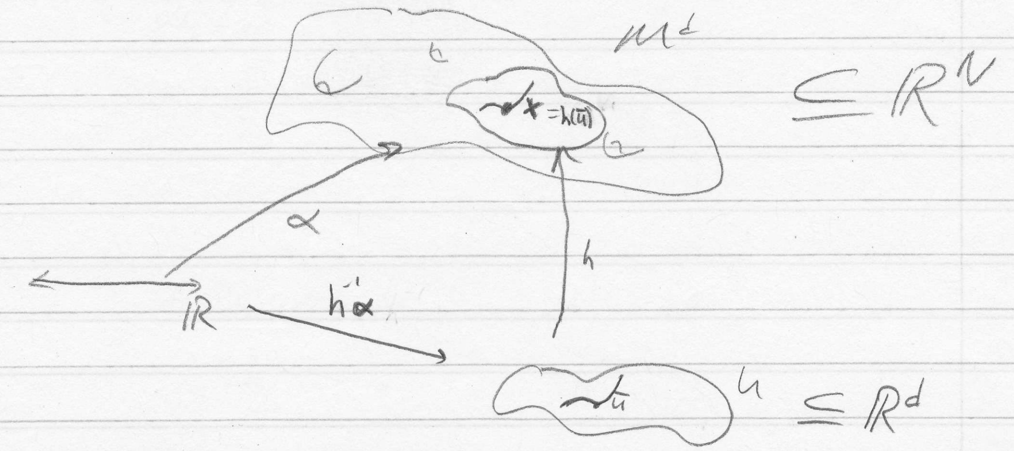
$$T_x M^d = \mathrm{Span}\left(\frac{\partial h}{\partial u_1}(\bar{u}), \dots, \frac{\partial h}{\partial u_d}(\bar{u})\right).$$

<u>Proof</u>: For any $\sum_{i=1}^{d} c_i \frac{\partial h}{\partial u_i}(\bar{u}) \in \mathrm{Span}\left(\frac{\partial h}{\partial u_i}(\bar{u})\right)$, there is a line $\alpha(t) = \bar{u} + t\begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix}$ $(\alpha : \mathbb{R} \to \mathbb{R}^d)$ such that $(h \circ \alpha)'(0) = (Dh)_{\bar{u}} \alpha'(0) = \sum_{i=1}^{d} c_i \frac{\partial h}{\partial u_i}(\bar{u})$:



In the other direction, if $\alpha : \mathbb{R} \to M^d$ is any curve with $\alpha(t) = x$, then $\alpha'(t) = [h(h^{-1}\alpha)]'(t) = (Dh)_{\bar{u}}(h^{-1}\alpha)'(t) \in \mathrm{Span}\left(\frac{\partial h}{\partial u_i}(\bar{u})\right)$. $\square$

The picture is:



<u>Def'n</u>: The <u>tangent</u> <u>bundle</u> to a smooth manifold $M^d \subseteq \mathbb{R}^N$ is the union of all tangent spaces $T_x M^d$, $x \in M^d$:

$$T(M^d) = \left\{ (x, \vec{v}) \in M^d \times \mathbb{R}^N \,\middle|\, \vec{v} \in T_x M^d \right\}.$$

We topologize $T(M^d)$ as a subspace of $M^d \times \mathbb{R}^N$, so that the <u>projection</u> map $\begin{matrix} T(M^d) \\ \pi \downarrow \\ M^d \end{matrix} \quad \begin{matrix} (x, \vec{v}) \\ \downarrow \\ x \end{matrix}$ is continuous.

<u>Lemma</u>: $T(M^d)$ is <u>locally trivial</u>: that is, we have homeomorphisms $\pi^{-1}(V) \xrightarrow{\varphi} V \times \mathbb{R}^d$ whenever $V \subseteq h(U)$ for some local parametrization $h: U \to M^d$. Moreover, these homeomorphisms are linear on each <u>fiber</u>, i.e. $\varphi|_{T_x(M^d)} \longrightarrow \{x\} \times \mathbb{R}^d$ is linear.

[Note: φ is a homeomorphism, so linear on fibers $\Longrightarrow$ isom. on fibers.]

<u>Proof</u>: The parametrization h gives us a choice of basis $\left\{ \frac{\partial h}{\partial u_j}(h^{-1}x) \right\}_{j=1}^{d}$ for $T_x(M^d)$. We now define

$$\varphi\left(x, \sum c_i \frac{\partial h}{\partial u_i}(h^{-1}x)\right) = \left(x, \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix}\right) \in V \times \mathbb{R}^d;$$

and $\varphi^{-1}\left(x, \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix}\right) = \left(x, \sum c_i \frac{\partial h}{\partial u_i}(h^{-1}x)\right)$.

These maps are well-defined and inverse to one another. Since $\frac{\partial h}{\partial u_j}$ is continuous, we see that φ^{-1} is continuous. To check continuity of φ, we use some basic linear algebra and Kramer's Rule. We will show that the injective linear maps $(Dh)_{h^{-1}x}$ have continuously varying left inverses L_x. We claim, then, that $\varphi(x, \vec{w}) = (x, L_x \vec{w})$, which is continuous. Indeed, we can write $\vec{w} = \sum c_i \frac{\partial h}{\partial u_i}(h^{-1}x) = (Dh)_{h^{-1}x}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ for some $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, and then $(x, L_x \vec{w}) = \left(x, L_x (Dh)_{h^{-1}x}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}\right) = \left(x, \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}\right)$, which was our definition of $\varphi(x, \vec{w})$.

Now, say $A : \mathbb{R}^d \hookrightarrow \mathbb{R}^n$ is a linear injection. Let $a_i = Ae_i$ and find $\vec{a}_{d+1}, \dots, \vec{a}_n \in \mathbb{R}^n$ such that $\{\vec{a}_1, \dots, \vec{a}_n\} \subseteq \mathbb{R}^n$ is a basis for $\mathbb{R}^n$. Then we have a linear map $\mathbb{R}^n \xrightarrow{L} \mathbb{R}^d$ which sends $\sum c_i \vec{a}_i$ to $\begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix}$, and $LA = I_d$ because

$$L\left(A\begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix}\right) = L\left(\sum_{i=1}^{d} c_i a_i\right) = \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix}.$$

We claim that the matrix of L is

$$d\left\{\underbrace{\left(\begin{array}{ccc|ccc} 1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & 0 & \vdots \\ 0 & & 1 & 0 & \cdots & 0 \end{array}\right)}_{n}\right. \cdot \left(\vec{a}_1 \cdots \vec{a}_n\right)^{-1} = P$$

Since $(\vec{a}_1 \cdots \vec{a}_n)\vec{e}_i = \vec{a}_i$, we have $(\vec{a}_1 \cdots \vec{a}_n)^{-1}\vec{a}_i = e_i$, and hence $P\vec{a}_i = \begin{cases} e_i, & i \le d \\ 0, & \text{else} \end{cases}$. This agrees with our definition of $L(a_i)$, so P and L agree on a basis.

Now, the Formula for P shows that P varies continuously with the vectors $\vec{a}_1, \cdots, \vec{a}_n$, because Kramer's rule gives a continuous formula for the inverse of a matrix. We must show that if the vectors $\{\vec{a}_1, \cdots, \vec{a}_d\}$ are varying continuously, we can choose the remaining vectors $\{\vec{a}_{d+1}, \cdots, \vec{a}_n\}$ continuously. But since $\det(\vec{a}_1 \cdots \vec{a}_n) \neq 0$, there exist open nbhds around each $\vec{a}_1, \cdots, \vec{a}_d$ so that for $\vec{a}_1', \cdots, \vec{a}_d'$ in these nbhds, $\det(\vec{a}_1' \cdots a_d' \; \vec{a}_{d+1} \cdots \vec{a}_n) \neq 0$, and hence $\{\vec{a}_1', \cdots, \vec{a}_d', \vec{a}_{d+1}, \cdots, \vec{a}_n\}$ is still a basis. So in fact, the <u>same</u> choice of $\vec{a}_{d+1} \cdots \vec{a}_n$ will work for nearby $\{\vec{a}_1', \cdots, \vec{a}_d'\}$. This allows us to choose (locally) left inverses to $(Dh)_{h^{-1}x}$ which vary continuously. This shows that φ is indeed continuous.

The tangent bundle $T(M^d)$ is our first example of a <u>vector bundle</u>.

<u>Def'n</u>: A <u>vector bundle</u> $V \xrightarrow{\pi} B$ is a pair of spaces ("total space" V, "base space" B) (V, B), together with a continuous projection map $\pi: V \to B$, and a choice of a <u>real vector space structure</u> on each <u>fiber</u> $\pi^{-1}(b)$ $(b \in B)$. This structure must satisfy the <u>local triviality</u> condition:

For each $b \in B$ there is an open nbhd $b \in U \subset B$ and a homeomorphism $V|_U = \pi^{-1}(U) \xrightarrow{\varphi} U \times \mathbb{R}^n$ (for some n) which is linear on each fiber $\pi^{-1}(b)$, $b \in U$.

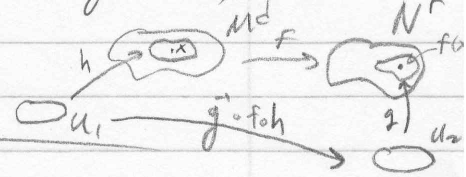
<u>Def'n</u> A <u>map</u> of vector bundles $(V, B) \to (V', B')$ is a comm. diag.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\pi"'] & V' \arrow[d, "\pi'"] \\ B \arrow[r, "\overline{\varphi}"] & B' \end{tikzcd}$$

such that φ restricts to a linear map on each fiber $\pi^{-1}(b)$.

<u>Smooth</u> <u>maps</u> b/w mflds give the first examples of maps between vector bundles:

<u>Def'n</u>: A smooth map $f: M^d \longrightarrow N^r$ b/w smooth mflds is a fcn such that for each $x \in M^d$ there exist parametrizations $h: U_1 \to M^d$, $g: U_2 \to N^r$ $(x \in h(U_1),\ f(x) \in g(U_2))$ such that $g^{-1} \circ f \circ h : U_1 \to U_2$ is smooth:



A smooth map $M^d \xrightarrow{f} N^r$ induces a map (its total derivative)

$$Df: TM^d \longrightarrow TN^r$$
$$\underset{\in T_{\alpha(t)}M^d}{\alpha'(t)} \longmapsto \underset{\in T_{f(\alpha(t))}N^r}{(f\circ\alpha)'(t)}$$

where $\alpha: (a,b) \longrightarrow M^d$ is any smooth curve. Note that with respect to the bases $\left\{\frac{\partial h}{\partial u_j}\right\}_1^d, \left\{\frac{\partial g}{\partial u_j}\right\}_1^r$, the restriction of Df to the fiber over $x \in M^d$ has matrix $\left(\frac{\partial (g^{-1}fh)_i}{\partial u_j}(h^{-1}x)\right)$. In particular, Df is linear on each fiber, hence a map of vector bundles.

<u>A Useful Fact</u>:

We call a map $\begin{array}{c} V \xrightarrow{\varphi} W \\ {\scriptstyle\pi}\searrow \ \swarrow{\scriptstyle\pi'} \\ B \end{array}$ of vector bundles over the same base an <u>isomorphism</u> if it is an isom. on each fiber.

<u>Lemma 2.3</u>: If φ is an isom. of v. bdles over B, then φ is a <u>homeomorphism</u>.

<u>Proof</u>: See M+S. The main point is that Cramer's Rule shows that the map $A \mapsto A^{-1}$ is a continuous map $GL_n\mathbb{R} \longrightarrow GL_n\mathbb{R}$. $\square$