Basic Examples: Spheres and Projective Spaces

Def'n: $S^{n-1} = \{\vec{x} \in \mathbb{R}^n \mid \Sigma x_i^2 = 1\}$ (the unit sphere).

Fact: S^{n-1} is a smooth mf'ld.

Proof: We can cover S^{n-1} by local parametrizations arising from projection onto hyperplanes:

$$\varphi_i^{\pm} : \underbrace{D^{n-1}}_{\{\vec{x} \in \mathbb{R}^{n-1} \mid \Sigma x_j^2 < 1\}} \longrightarrow S^{n-1}$$

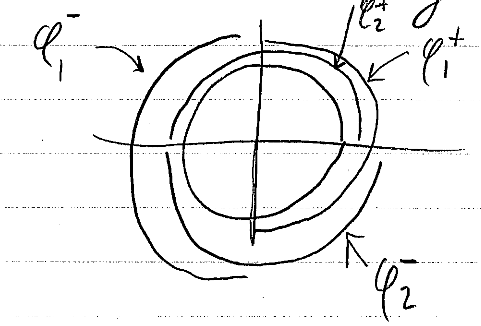
$$(x_1, \dots, x_{n-1}) \longmapsto (x_1, \dots, \underset{\uparrow\, i^{th} \text{ position}}{\pm\sqrt{1 - \Sigma x_j^2}}, \dots, x_{n-1}).$$

This gives $2n$ local parametrizations, since we can choose any $i \in \{1, \dots, n\}$ and either sign $+/-$.

These maps are homeomorphisms onto their images, because they are inverse to the relevant projections.

The transition fcns $(\varphi_j^{\pm})^{-1}\varphi_i^{\pm}$ are then clearly smooth (since $\sqrt{y}$ is smooth away from $y = 0$). $\square$

Example: $S^1 \subset \mathbb{R}^2$ is covered by 4 charts:

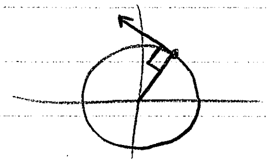


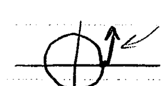
<u>Prop'n</u>: The tangent bundle to S^1 is trivial.

<u>Proof</u>: We have a continuous section $\underset{(x_1,x_2)\mapsto(-x_2,\,x_1)}{S^1 \longrightarrow T(S^1)}$.

One can check (using MS Lemma 1.2) that $(-x_2, x_1) \in T_{(x_1,x_2)} S^1$. □

Geometrically, this function just rotates (x_1, x_2) counter-clockwise by $90°$:



This can be obtained more systematically from complex multiplication in $\mathbb{R}^2 \cong \mathbb{C}$: starting with the tangent vector $(0,1) = i$ in $T_{(1,0)} S^1$ ( i)

we simply multiply by $e^{i\theta} = (\cos\theta, \sin\theta)$ to obtain a tangent vector at $e^{i\theta} = (\cos\theta, \sin\theta) \in \mathbb{C} = \mathbb{R}^2$:

$$\begin{aligned} ie^{i\theta} = i(\cos\theta + i\sin\theta) &= i\cos\theta - \sin\theta \\ &= (-\sin\theta, \cos\theta). \end{aligned}$$

So this process transports the vector $(0,1) \in T_{(1,0)} S^1$ to the vector $(-\sin\theta, \cos\theta) \in T_{(\cos\theta, \sin\theta)} S^1$. Letting $x_1 = \cos\theta$, $x_2 = \sin\theta$ gives the original formula.

This process really shows that any Lie group (= smooth mfld equipped with smooth mult'n + inverse maps) has trivial tangent bundle. (See MS p. 20 for the case of S^3, which is the group of unit quaternions).

<u>Def'n</u> (Real Projective Space)

$$\mathbb{R}P^n = S^n \Big/ {\scriptstyle \vec{x} \sim -\vec{x} \atop \text{for all } \vec{x} \in S^n}.$$

We equip $\mathbb{R}P^n$ with the quotient topology.

<u>Fact</u>: $\mathbb{R}P^n$ is a smooth mfld.

<u>Proof</u>: We can use the same charts as for S^n, except that we must compose with the projection $\pi: S^n \to \mathbb{R}P^n$. Note that $\varphi_i^{\pm}(D^n)$ never contains two antipodal points, so $\pi \circ \varphi_i^{\pm} : D^n \to \mathbb{R}P^n$ is still a homeomorphism onto its image. The transition maps are exactly the same as those for the sphere. $\square$

<u>The canonical line bdle over $\mathbb{R}P^n$</u>:

We have a line bdle γ_n^1 over $\mathbb{R}P^n$, whose fiber over $\{\vec{x}, -\vec{x}\} \in \mathbb{R}P^n$ consists of the line $\{c\vec{x} \mid c \in \mathbb{R}\} \subseteq \mathbb{R}^{n+1}$.

Formally, the total space of γ_n^1 is given by

$$E(\gamma_n^1) = \left\{ (\{\pm\vec{x}\}, \vec{v}) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \;\middle|\; \vec{v} = c\vec{x} \text{ for some } c \in \mathbb{R} \right\},$$

with the subspace topology. The projection is $\begin{aligned} E(\gamma_n^1) &\to \mathbb{R}P^n \\ (\{\pm x\}, v) &\mapsto \{\pm x\} \end{aligned}$

and each fiber has its natural vector space structure. It is easy to check (MS p. 16) that γ_n^1 is locally trivial (over the above coord nbhds for $\mathbb{R}P^n$, say).

<u>Theorem 2.1</u> γ_n^1 is not trivial, for any $n \geq 1$.

<u>Proof</u>: Trivial bundles $\begin{matrix} V \\ \pi\downarrow \\ X \end{matrix}$ admit nowhere-zero sections $\begin{matrix} V \\ s\uparrow\downarrow\pi \\ X \end{matrix}$ (with $\pi s = \mathrm{id}_X$). If γ_n^1 had such a section s, then

$$\begin{array}{ccccc} S^n & \xrightarrow{\pi} & P^n & \xrightarrow{s} & E(\gamma_n^1) \\ x & \longmapsto & \{\pm x\} & \longmapsto & (\{\pm x\}, c(x)\cdot x) \end{array}$$

gives a continuous function $c: S^n \longrightarrow \mathbb{R}-\{0\}$ with $c(x) = -c(-x)$, but this contradicts the Intermediate Value Theorem. $\square$

<u>Rmk</u>: Continuity of c, and the fact that the obvious local trivializations of γ_n^1 are homeomorphisms, are essentially the same and can be proven just as in the proof of local triviality for tangent bdles.