Clutching Functions & Principal Bundles

Vector bundles can be built up from trivial bundles $U \times \mathbb{R}^n$

via clutching functions:

Def'n: If $V \xrightarrow{\pi} B$ is a vector bundle and $\varphi_1 : \pi^{-1} U_1 \xrightarrow{\cong} U_1 \times \mathbb{R}^n$,

$\varphi_2 : \pi^{-1} U_2 \xrightarrow{\cong} U_2 \times \mathbb{R}^n$ are trivializations with $U_1 \cap U_2 \neq \emptyset$, then the isomorphism

$$\varphi_{21} := \varphi_2 \circ \varphi_1^{-1} : U_1 \cap U_2 \times \mathbb{R}^n \to U_1 \cap U_2 \times \mathbb{R}^n$$

is called a transition fcn for V. For each $x \in U_1 \cap U_2$,

$\varphi_{21} : \{x\} \times \mathbb{R}^n \longrightarrow \{x\} \times \mathbb{R}^n$ is a linear isomorphism, so

we may view φ_{21} as a mapping $U_1 \cap U_2 \to GL_n(\mathbb{R})$, called a clutching function.

Lemma: (the cocycle condition):

If $U_1 \cap U_2 \cap U_3 \neq \emptyset$ and $\varphi_i : \pi^{-1} U_i \xrightarrow{\cong} U_i \times \mathbb{R}^n$, then we

have $$\boxed{\varphi_{32} \varphi_{21} = \varphi_{31}}$$ (either as comp. of fcns or as products of matrices)

Proof: $\varphi_{32} \circ \varphi_{21} = (\varphi_3 \varphi_2^{-1})(\varphi_2 \circ \varphi_1^{-1}) = \varphi_3 \circ \varphi_1^{-1} = \varphi_{31}$. $\square$

We can now construct v. bdles using clutching fcns:

Prop'n: If $B = \bigcup_{i \in I} U_i$ (open sets) and $\varphi_{ji} : U_i \cap U_j \to GL_n \mathbb{R}$ are

clutching fcns satisfying the cocycle condition, then

$$V = \left(\coprod_{i \in I} U_i \times \mathbb{R}^n \right) \Big/ (u, \vec{v}) \sim (u, \varphi_{ji} \vec{v}) \text{ for all } u \in U_i,\ i \in I$$

is a v. bdle over B w/ clutching fcns φ_{ji}.

Principal Bundles

When we have a vector bdle described in terms of clutching fcns, we can get rid of $\mathbb{R}^n$ entirely:

Say $\underset{B}{\overset{V}{\downarrow}}$ has clutching fcns $\varphi_{ji} : U_i \cap U_j \to GL_n\mathbb{R}$.
($\{U_i\}$ an open cover of B)

Then we can use these fcns to construct a bundle whose <u>fibers</u> are $GL_n\mathbb{R}$ itself:

$$P = P_V = \left(\coprod_i U_i \times GL_n\mathbb{R} \right) \Big/ \sim$$

where $\underset{U_i \times GL_n\mathbb{R}}{(u, A)} \sim \underset{U_j \times GL_n\mathbb{R}}{(u, \varphi_{ji}(u)A)}$ if $u \in U_i \cap U_j$, $A \in GL_n\mathbb{R}$.

[We only glue pts from different elements of the <u>disjoint union</u>.]

The projections $U_i \times GL_n\mathbb{R} \to U_i \hookrightarrow B$ respect the equivalence reln, and yield a continuous projection map $P \xrightarrow{\pi} B$. Each fiber of this map is <u>non</u> <u>canonically</u> homeomorphic to $GL_n\mathbb{R}$, and in fact $\pi^{-1}(U_i) \cong U_i \times GL_n\mathbb{R}$ for each i.

We can recover V by <u>mixing</u>: the space V admits a well-defined <u>right</u> action of $GL_n\mathbb{R}$, given by

$$(u, A) \cdot B = (u, AB).$$

<u>Lemma:</u> $P_V \times_{GL_n\mathbb{R}} \mathbb{R}^n \cong V$ as vector bdles over B.

Here $P_V \times_{GL_n\mathbb{R}} \mathbb{R}^n := (P_V \times \mathbb{R}^n) \Big/ \begin{pmatrix} (p,x) \sim (pA, A^{-1}x) \\ \text{for all } p \in P, x \in \mathbb{R}^n, A \in GL_n\mathbb{R} \end{pmatrix}$.

<u>Proof:</u> We have a map

$$\begin{aligned} P \times \mathbb{R}^n &\longrightarrow V = \left(\coprod_i U_i \times \mathbb{R}^n\right)\Big/\sim \;, \\ ([U_i, A], x) &\longmapsto [U_i, Ax] \end{aligned}$$

which is continuous and factors through the quotient on the left. On each fiber, it can be identified with the linear isomorphism $x \mapsto Ax$, and its inverse is given by the continuous map

$$\begin{aligned} P \times_{GL_n\mathbb{R}} \mathbb{R}^n &\longleftarrow \coprod_i U_i \times \mathbb{R}^n \\ [[U_i, I], x] &\longleftarrow (U_i, x) \;, \end{aligned}$$

which factors through the equivalence reln on the right. $\square$

Note that the vector space structure on $(P \times_{GL_n\mathbb{R}} \mathbb{R}^n)|_b$ is just that inherited from $\mathbb{R}^n$.

These constructions allow us to pass between vector bundles and principal $GL_n\mathbb{R}$-bundles. To make the correspondence complete, we need a notion of <u>maps</u> b/w principal $GL_n\mathbb{R}$ bundles:

If $\begin{array}{c} P_1 \\ \downarrow \pi_1 \\ B_1 \end{array}$ and $\begin{array}{c} P_2 \\ \downarrow \pi_2 \\ B_2 \end{array}$ are principal $GL_n\mathbb{R}$-bundles,

then a <u>map</u> $P_1 \to P_2$ consists of a commutative diagram

$$\begin{tikzcd} P_1 \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi_1"] & P_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "\varphi"] & B_2 \end{tikzcd}$$

in which $\tilde{\varphi}$ is <u>equivariant</u>: $GL_n\mathbb{R}$ acts on both P_1 and P_2, and $\tilde{\varphi}$ must satisfy

$$\tilde{\varphi}(p \cdot A) = \tilde{\varphi}(p) \cdot A$$

for all $p \in P_1$ and $A \in GL_n\mathbb{R}$.

<u>Exercise</u>: Maps b/w vector bdles induce maps b/w the associated $GL_n\mathbb{R}$-bundles, and vice-versa, and these correspondences respect composition of maps. Hence in particular, when applied to the identity map $V \xrightarrow{id_V} V$ we see that different local trivializations produce the same $GL_n\mathbb{R}$ bundle up to isomorphism.

Metrics and Principal $O(n)$-bundles:

Recall that an inner product $\langle , \rangle$ on a real v.sp. V is a symmetric, bilinear function $V \times V \longrightarrow \mathbb{R}$, $v, w \longmapsto \langle v,w \rangle$ which is positive definite, i.e. $\langle v,v\rangle > 0$ for $v \neq 0$.

We want to consider vector bundles $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ equipped with a metric on each fiber.

Def'n: A Euclidean vector bundle is a real v. bdle $\begin{smallmatrix} E \\ \downarrow \pi \\ B \end{smallmatrix}$ together w/ a continuous function

$$E \times_B E \xrightarrow{\langle , \rangle} \mathbb{R}$$
$$E \times_B E = \{ (v,w) \in E \times E \mid \pi(v) = \pi(w) \}$$

whose restriction to each subset $\pi^{-1}(b) \times \pi^{-1}(b) \subseteq E \times_B E$ $(b \in B)$ is an inner product on $\pi^{-1}(b)$.

Alternate Viewpoint: An inner product on a real v. space V is equivalent to a positive definite quadratic function $\mu : V \to \mathbb{R}$. This means $\mu(v) > 0$ for $v \neq 0$, and $\mu(v) = \sum l_i(v)\, l_i'(v)$ for some linear functions $l_i, l_i' : V \to \mathbb{R}$.

Given μ, we obtain an inner product by the "polarization" formula
$$\langle v,w \rangle = \tfrac{1}{2}\left(\mu(v+w) - \mu(v) - \mu(w)\right)$$
and given $\langle , \rangle$ we obtain the associated μ by setting
$$\mu(v) = \langle v,v \rangle =: |v|^2$$

Note that if we express v in an o.n. ("orthonormal") basis $\{e_i\}$, then

$$\mu(v) = \langle \Sigma \lambda_i e_i, \Sigma \lambda_i e_i \rangle = \sum_{i,j} \lambda_i \lambda_j,$$

and the functions $v \mapsto \lambda_i$ are all linear.

Now continuity of $\langle , \rangle : E \times_B E \to \mathbb{R}$ is equivalent to continuity of the associated $\mu : E \to \mathbb{R}$.

Lecture 3 A Euclidean bundle is not only locally isomorphic to a trivial bundle, it is also automatically locally isometric to a trivial bundle with its standard inner product. This is Lemma 2.4 in MS, and follows from continuity of the Gram-Schmidt orthogonalization process.

Clutching Functions for Euclidean Bundles:

Proposition: Let $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ be a Euclidean v. bdle. Then there exist local trivializations $\varphi_i : U_i \times \mathbb{R}^n \to E$ such that the associated clutching functions $\varphi_{ij} : U_i \cap U_j \to GL_n\mathbb{R}$ all land inside $O(n) = \{A \in GL_n\mathbb{R} \mid AA^T = I_n\}$.

<u>Proof</u>: Simply choose φ_i to be the local isometries guaranteed by Lemma 2.4 (MS). $\square$

This proposition allows us to associate an $O(n)$-bdle to each Euclidean v. bdle, just as we associated a $GL_n\mathbb{R}$-bdle to each ordinary v. bdle. Again, this bdle depends only on the isometry type of the Euclidean bdle.

In the other direction, the mixed bdle

$$P \times_{O(n)} \mathbb{R}^n \quad \text{associated to an } O(n)\text{-bdle } \begin{array}{c} P \\ \downarrow \\ B \end{array}$$
$$\downarrow$$
$$B$$

inherits a metric from $\mathbb{R}^n$, because the transition fcns are isometries.

<u>Remark</u>: All of this works equally well with $\mathbb{R}$ replaced by $\mathbb{C}$. Metrics on complex bundles are required to be Hermitian, that is they are conjugate-linear in the 2nd coord: $\langle v, zw\rangle = \bar{z}\langle v, w\rangle$ for $z \in \mathbb{C}$. The transition fcns then lie in $U(n) = \{A \in GL_n\mathbb{C} \mid A\bar{A}^T = I_n\}$, so cplx Hermitian bdles correspond to (principal) $U(n)$-bundles.

The $GL_n\mathbb{R} / O(n)$ bdles we have associated to vector / Euclidean bdles are examples of the general notion of <u>principal bundles</u>:

<u>Def'n</u> Let G be a topological group (i.e. the mult'n map $G \times G \to G$ and inversion $G \to G$ are continuous).

A (right) <u>principal</u> <u>G-bdle</u> over a space B is a map

$$\begin{array}{c} P \\ \downarrow{\scriptstyle p} \\ B \end{array}$$

together with an open covering $\{U_i\}_{i \in I}$ of B and homeomorphisms $\varphi_i : U_i \times G \xrightarrow{\cong} p^{-1}(U_i)$ satisfying:

1) The "local trivializations" φ_i respect the projections to B: that is, $p \circ \varphi_i = \pi_1$, where $\pi_1 : U_i \times G \to U_i$ is proj'n on the first coordinate.

2) (Principality) Whenever $U_i \cap U_j \neq \emptyset$, the composite

$$\varphi_{ji} = \varphi_j^{-1} \varphi_i : U_i \cap U_j \times G \longrightarrow U_i \cap U_j \times G$$

has the form $\varphi_{ji}(u, g) = (u, \tau_{ji}(u) \cdot g)$ for some $\tau_{ji}(u) \in G$

[here $\tau_{ji}(u)$ depends only on u, not on g].

<u>Remark</u>: The function $u \mapsto \tau_{ji}(u)$ is automatically continuous, b/c $\tau_{ji}(u) = \pi_2(\varphi_{ji}(u,g)) \cdot (\pi_2(u,g))^{-1}$ (where $\pi_2 : U_i \cap U_j \times G \to G$ is projection onto G).

The reason for calling this a right principal bdle is:

<u>Lemma</u>: If $\begin{smallmatrix} P \\ \downarrow p \\ B \end{smallmatrix}$ is a (right) principal G-bdle, then P admits a continuous right action $P \times G \to P$ such that

1) The quotient P/G is homeomorphic to B
2) The trivializations $\varphi_i : U_i \times G \to p^{-1}(U_i)$ are G-equivariant (where $(u,g)\cdot h := (u, gh)$).

[Note that 2) implies that G acts freely, and acts transitively on each fiber $p^{-1}(b)$.]

<u>Proof</u>: We transport the action
$$(U_i \times G) \times G \longrightarrow U_i \times G$$
$$(u_i, g), h \longmapsto u_i, gh$$
to P using the local trivializations $\varphi_i : U_i \times G \xrightarrow{\cong} p^{-1}(U_i)$:

For $\in P$, $g \in G$ we define
$$x \cdot g = \varphi_i(\varphi_i^{-1}(x) \cdot g).$$

This is well-defined by principality: if $u = p(x) \in U_i \cap U_j$, we must check that $\varphi_i(\varphi_i^{-1}(x) \cdot g) = \varphi_j(\varphi_j^{-1}(x) \cdot g)$, i.e. that $\varphi_j^{-1} \varphi_i(\varphi_i^{-1}(x) \cdot g) = \varphi_j^{-1}(x) \cdot g$.

Letting $\varphi_i^{-1}(x) = (u, h)$, we have
$$\varphi_j^{-1}\varphi_i(\varphi_i^{-1}(x)\cdot g) = \varphi_{ji}((u,h)\cdot g) = \varphi_{ji}(u, hg)$$
$$= (u, \tau_i(u) hg) = (u, \tau_i(u)h) \cdot g$$
$$= \varphi_{ji}(u,h) \cdot g = \varphi_j^{-1}(\varphi_i(u,h)) \cdot g$$
$$= \varphi_j^{-1}(x) \cdot g.$$

To see that $P/G \cong B$, note that we have a comm. diagram

$$\begin{tikzcd} & P \arrow[dl, "q"'] \arrow[d, "p"] \\ P/G \arrow[r, dashed, "f"] & B \end{tikzcd}$$

in which f is a continuous bijection. To see that f is an open map, consider any open set $\bar{V} \subseteq P/G$. Then $V = q^{-1}(\bar{V})$ is open in P, and $f(\bar{V}) = p(q^{-1}\bar{V})$. But p is an open map, b/c locally it is just the projection $U_i \times G \longrightarrow U_i$. $\square$

Basic Examples:

The $GL_n\mathbb{R} / O(n)$-bdle associated to a vector/Euclidean bdle are principal bundles. In fact, for any group G and any clutching data $\varphi_{ji} : U_i \cap U_j \longrightarrow G$ ($\{U_i\}_i$ an open cover of some base B), the bundle

$$P = \left(\coprod_i U_i \times G \right) \Big/ (u,g) \sim (u, \varphi_{ji}(u) g)$$

$$\downarrow p$$

$$B$$

is principal, w/ local trivializations the inclusions $U_i \times G \hookrightarrow P$. The associated action is just $[u,g] \cdot h = [u, gh]$.

Note: The fact that $U_i \times G \hookrightarrow P$ is a homeomorphism onto its image $p^{-1}(U_i)$ follows from the fact that

$$\begin{array}{ccc} U_i \cap U_j \times G & \longrightarrow & U_i \cap U_j \times G \\ (u, g) & \longmapsto & (u, \varphi_{ji}(u) g) \end{array}$$

is a homeomorphism.

Maps b/w Principal Bdles:

Def'n: If $\begin{array}{c} P_1 \\ \downarrow p_1 \\ B_1 \end{array}$ and $\begin{array}{c} P_2 \\ \downarrow p_2 \\ B_2 \end{array}$ are principal G-bdles, a map from $P_1 \to P_2$ is a diagram

$$\begin{tikzcd} P_1 \arrow[r, "\varphi"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\circlearrowleft"] & P_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "\bar{\varphi}"] & B_2 \end{tikzcd}$$

$\boxed{\text{indicates commutativity}}$

<u>in which φ is</u> G-equivariant.

We've seen that fiber-wise isomorphisms of V. bdles are honest isomorphisms; here is the analogue for principal bdles.

Prop'n: If $\begin{tikzcd} P_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & P_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$ is a map of principal G-bdles (covering Id_B) then φ is a homeomorphism (and its inverse $\varphi^{-1}: P_2 \to P_1$ is also a map of principal bdles).

Pf: Locally, φ has the form $\begin{array}{ccc} U \times G & \xrightarrow{\varphi} & U \times G \\ (u, g) & \longmapsto & (u, \varphi_2(u,g)) \end{array}$, where $g \mapsto \varphi_2(u,g)$ is G-equivariant (wrt right mult. in G). This means $\varphi_2(u,g) = h(u) g$

where $h(u) := \varphi_2(u,g)g^{-1}$. Hence $h : U \to G$ is continuous,

and now $\varphi^{-1} : U \times G \to U \times G$ is the continuous map $(u,g) \mapsto (u, h(u)^{-1}g)$.

So φ is a continuous bijection, and its inverse is continuous. □

<u>Corollary</u>: If $\begin{array}{c} P \\ \downarrow p \\ B \end{array}$ is a principal G-bdle admitting a continuous section $\begin{array}{c} P \\ p\downarrow\uparrow s \\ B \end{array}$ $(ps = 1d_B)$ then P is trivial, i.e. there is an isom. of G-bdles $\begin{tikzcd} P \arrow[rr, "\cong"] \arrow[dr] & & B \times G \arrow[dl] \\ & B & \end{tikzcd}$.

<u>Pf</u>: Define $\varphi : \begin{array}{l} B \times G \longrightarrow P \\ (b,g) \longmapsto s(b)\cdot g \end{array}$, and apply the Prop'n. □

Here is another application of the Prop'n:

<u>Exercise</u>: Say $\begin{array}{c} V \\ \downarrow \\ B \end{array}$ is a v. bdle, and say $\{U_i, \varphi_i\}_i$, $\{V_j, \psi_j\}_j$ are two different local trivializations of V. Then the associated principal $GL_n\mathbb{R}$ bdles for these different clutching data are isomorphic.
