Classification of Principal Bundles

Pullbacks: Say $f: X \to B$ is a map, and $\begin{matrix} E \\ \downarrow p \\ B \end{matrix}$ is a bdle (e.g. a vector bdle, a Eucl. bdle, or a princ. bdle). Then $f^*(E) = \{(x,e) \in X \times E : p(e) = f(x)\}$ is also a bdle, of the same type.

$$\begin{matrix} f^*(E) & & (x,e) \\ \downarrow & & \downarrow \\ X & & x \end{matrix}$$

[If E is trivial over $U \subset B$, $f^* E$ is trivial over $f^{-1}(U)$.]

Our next goal is the following theorem, which describes the set $\mathrm{Prin}_G(X) = \{\text{principal } G\text{-bdles over } X\}/\cong$ of isom. classes of G-bdles homotopically.

Theorem: If $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is a principal G-bdle such that all htpy groups $\pi_*(E)$ are trivial, then for every CW cplx X, the map

$$\begin{aligned} \mathrm{Map}(X, B) &\xrightarrow{\Phi} \mathrm{Prin}_G(X) \\ f: X \to B &\longmapsto [f^*(E)] \end{aligned}$$

factors through homotopy classes and gives a bijection

$$[X, B] \xrightarrow{\cong} \mathrm{Prin}_G(X).$$

Notation/Terminology: The bdle $E \to B$ is called a universal principal G-bdle. One often denotes the base space B by BG and the total space E by EG; BG is called a classifying space for G.

The proof of this theorem will require several important ideas, constructions and results.

We begin by considering surjectivity of Φ.

Lemma: Any map of principal G-bdles
$$\begin{tikzcd} P_1 \arrow[r, "\tilde{\varphi}"] \arrow[d, "p_1"] & P_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "\varphi"] & B_2 \end{tikzcd}$$
induces an isomorphism $P_1 \xrightarrow{\cong} \varphi^* P_2$.

Pf: We have a map $\begin{array}{l} P_1 \longrightarrow \varphi^* P_2 \\ x \longmapsto (p_1(x), \tilde{\varphi}(x)) \end{array}$ which is equivariant and covers Id_{B_1}. The result now follows from the Prop'n. $\square$

To prove that every bdle $\begin{array}{c} P \\ \downarrow \\ X \end{array}$ over a CW cplx is pulled back from the universal bdle $\begin{array}{c} EG \\ \downarrow \\ BG \end{array}$, we just need to construct an equivariant map $P \xrightarrow{\tilde{\varphi}} EG$. Since $X = P/G$, the diagram
$$\begin{tikzcd} P \arrow[r, "\tilde{\varphi}"] \arrow[d] & EG \arrow[d] \\ X = P/G \arrow[r, dashed, "\varphi"] & BG = EG/G \end{tikzcd}$$
can always be filled in.

Key Lemma (Ehresman): Given a principal G-bdle $\begin{array}{c} P \\ \downarrow p \\ X \end{array}$ and a G-space E, there is a bijection between G-equivariant maps $P \to E$ and sections of the mixed bundle
$$\begin{array}{c} P \times_G E = (P \times E)/(p,e) \sim (pg, eg) \\ \downarrow \\ X \end{array}.$$

Proof: Given a G-map $P \xrightarrow{\varphi} E$, we have a diagram:
$$\begin{tikzcd} P \arrow[r, "(Id, \varphi)"] \arrow[d] & P \times E \arrow[d, "\pi"] \\ X = P/G \arrow[r, dashed, "s"] & P \times_G E \end{tikzcd}.$$
Equivariance of φ implies that $\pi \circ (Id, \varphi)$ factors through P/G, and s is the desired section.

In the other direction, given $\begin{smallmatrix} P\times_G E \\ s\uparrow\downarrow\pi \\ P/G = X \end{smallmatrix}$, we define $P \xrightarrow{\varphi} E$ as follows:

We can always write $s([p]) = [p, e]$ for some (unique) $e \in E$ and we set $\varphi(p) = e$. Continuity of φ can be checked locally, using the fact that

$$\begin{aligned} (U\times G)\times_G E &\xrightarrow{\cong} U\times E \\ [u,g,e] &\longmapsto (u, eg^{-1}) \\ [u,1,e] &\longleftarrow\!\shortmid (u,e) \end{aligned}$$

. $\square$

Lecture 4: 1

Prop'n: Let X be a CW cplx, and let $\begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix}$ be a fiber bundle with fiber F. If $\pi_* F = 0$ for $* \le \dim X$ then E admits a section (if $\dim X = \infty$, we just require $\pi_* F = 0$ for $* \geq 0$).

In particular, the mixed bdles $\begin{smallmatrix} P\times_G EG \\ \downarrow \\ X \end{smallmatrix}$ always admit sections, so we obtain the required G-map $P \to EG$.

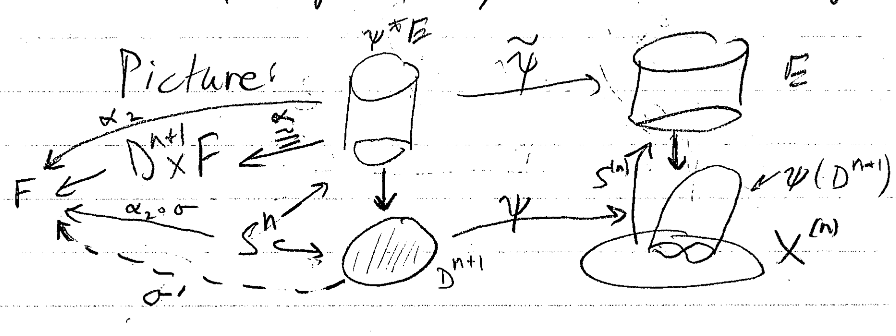
In the proof, we will need to use the fact that every principal G-bdle over a contractible space is trivial. This follows from the Bundle Htpy Theorem, which we'll prove next time.

<u>Proof of Prop'n:</u> By induction on skeleta. Certainly we can define a section $s \curvearrowright \begin{smallmatrix} E \\ \downarrow \\ X^{(0)} \end{smallmatrix}$. Now assume we have a section $s^{(n)}$ defined on $X^{(n)}$; we must extend it over each $(n+1)$-cell. Let $\psi : D^{n+1} \to X$ be the characteristic map for an $(n+1)$-cell in X, so $\psi|_{S^n} : S^n \to X^{(n)}$. Then since $D^{n+1} \simeq *$, the pullback $\psi^* E$ is a trivial bdle over D^{n+1}. The composite $S^n \xrightarrow{\psi} X^{(n)} \xrightarrow{s^{(n)}} E$ gives us a section of $\sigma \curvearrowright \begin{smallmatrix} \psi^* E \\ \downarrow \\ S^n \end{smallmatrix}$: $\sigma(x) = (x, s^{(n)}\psi(x))$. Since $\psi^* E \xrightarrow[\alpha]{\cong} D^{n+1} \times F$, we get a corresponding map $\alpha_2 \circ \sigma : S^n \to F$ which extends to $\sigma' : D^{n+1} \to F$ (b/c $\pi_{n+1} F = 0$). Letting $\tilde{\sigma} : D^{n+1} \to E$ be the map $\tilde{\sigma}(x) = \tilde{\psi}\alpha^{-1}(x, \sigma'(x))$ (where $\begin{array}{ccc} \psi^* E & \xrightarrow{\tilde{\psi}} & E \\ \downarrow & & \downarrow \\ D^{n+1} & \xrightarrow{\psi} & X \end{array}$) the desired exten of $s^{(n)}$ is:

$$D^{n+1} \cup_{\psi} X^{(n)} \xrightarrow{\tilde{\sigma} \cup s^{(n)}} E$$

(tracing the defns shows that that $\tilde{\sigma}(x) = s^{(n)}\psi(x)$ for $x \in S^n$, so this is well-defined and continuous).

Picture:



$\square$

A key tool for studying bundles is the following result:

<u>The Bundle Homotopy Theorem:</u> Let B be a paracompact Hausdorff space, and let $\begin{matrix} E \\ \downarrow p \\ B\times I \end{matrix}$ be a principal G-bdle. Then there is a bundle isomorphism

$$\begin{tikzcd} E \arrow[rr, "\cong"] \arrow[dr] & & E_0 \times I \arrow[dl] \\ & B\times I & \end{tikzcd}$$

where $E_0 = p^{-1}(B\times\{0\})$.

[The corresponding result for vector bdles/Euclidean bdles follows by clutching/mixing.]

<u>Rmk</u>: This isomorphism restricts to give isomorphisms $E_t \cong E_0$ for all $t \in I$, where $E_t = p^{-1}(B\times\{t\})$. In particular, $E_0 \cong E_1$, so we can think of the BHT as saying that "homotopic bdles" are isomorphic.

<u>Corollary 1:</u> If $f_0, f_1 : X \to Y$ are homotopic and $\begin{matrix} E \\ \downarrow \\ Y \end{matrix}$ is a principal G-bdle, then there is an isomorphism

$$\begin{tikzcd} f_0^*E \arrow[rr, "\cong"] \arrow[dr] & & f_1^*E \arrow[dl] \\ & X & \end{tikzcd}.$$

<u>Pf</u>: Any homotopy $H: X\times I \to Y$ $(H_0 = f_0,\ H_1 = f_1)$ induces a bundle htpy $\begin{matrix} H^*E \\ \downarrow \\ X\times I \end{matrix}$ connecting f_0^*E to f_1^*E. $\square$

This corollary tells us that we have a well-defined factorization

$$f \longmapsto f^*E$$

$$\begin{tikzcd} \mathrm{Map}(X,B) \arrow[rr] \arrow[dr] & & \mathrm{Prin}_G(X) \\ & {[X,B]} \arrow[ur] & \end{tikzcd}$$

for <u>any</u> paracpt Hausd. space X and <u>any</u> bdle $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$.

<u>Corollary 2:</u> If $F: X \xrightarrow{\simeq} Y$ is a htpy equiv. b/w paracpt. Hausd. spaces, then F induces a <u>bijection</u> $f^*: \mathrm{Prin}_G Y \to \mathrm{Prin}_G X$.

<u>Pf</u>: If $g: Y \to X$ is a htpy inverse ~~to~~ f, then $f^*g^* \simeq \mathrm{Id}_Y$, $g^*f^* \simeq \mathrm{Id}_X$. So $g^*f^*(E) \cong (\mathrm{Id}_X)^*E = E$ and $f^*g^* E \cong (\mathrm{Id}_Y)^*E = E$. $\square$

In particular, this shows that all bdles over a contractible (paracpt., Hausd.) space are trivial; we used this already in the case where the base is a disk.

We can now complete the proof of our theorem on universal bdles:

<u>Proof</u>: We want to show that $\mathrm{Map}(X,B) \longrightarrow \mathrm{Prin}_G(B)$, $f \longmapsto f^*E$ induces a <u>bijection</u> $[X, B] \xrightarrow{\cong} \mathrm{Prin}_G(B)$ when X is CW and $\pi_* E = 0$. This map is well-defined by the BHT, [Note: every CW cplx is paracompact, by a thm of Miyazaki] and we've proven that it's surjective.

To prove injectivity, we must show that if $f^*E \cong g^*E$ for some $f, g: X \to B$, then $f \simeq g$. This is just a "relative" form of the surjectivity statement, and we prove it the same way: letting $P = f^*E$, we have diagrams

$$\begin{tikzcd} P \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d] \\ X \arrow[r, "f"] & B \end{tikzcd}, \quad \begin{tikzcd} P \arrow[r, "\tilde{g}"] \arrow[d] & E \arrow[d] \\ X \arrow[r, "g"] & B \end{tikzcd}$$

which (by the Key Lemma) correspond to sections s_f, s_g of $P \times_G E$.

These give us a partially defined section of $\begin{array}{c} P\times_G(E\times I) \\ \downarrow \\ X\times I \end{array}$ (defined on $X\times\{0,1\}$) and just as before we can extend this section over the rest of the CW cplx $X\times I$. This gives a section $s: X\times I \longrightarrow P\times_G(E\times I)$, $(x,t)\longmapsto[\tilde{x},e,t]$ which translates under the Key Lemma to a map $F: X\times I \longrightarrow B\times I$, $(x,t)\longmapsto[s_x(t)]$, and the composite $X\times I \xrightarrow{F} B\times I \longrightarrow B$ is the desired htpy b/w $f,g: X\to B$. $\square$

Porism: If $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ is a principal G-bdl w/ $\pi_* E = 0$ for $*\leq n$, then $[X,B]\longrightarrow \mathrm{Prin}_G(X)$, $f\longmapsto f^*E$ is bijective for CW cplx's X of dim'n $< n$, surjective for dim $X = n$.
(We call E "$(n-1)$-universal".)

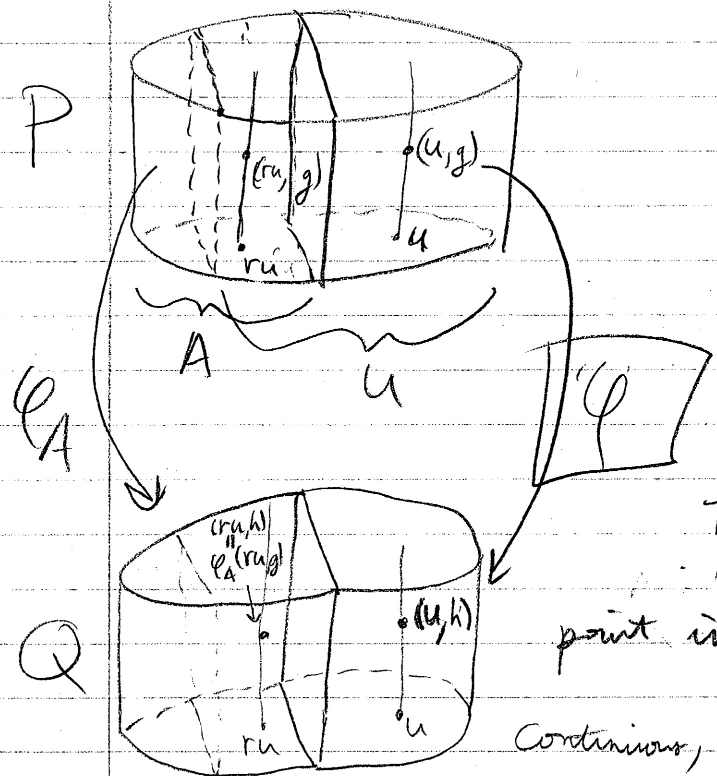
Pf: Just follow the previous proof, noting that $\dim(X\times I) = n+1$, so we don't get injectivity if $\pi_{n+1}E\neq 0$. $\square$

To prove the BHT, we need two lemmas:

Lemma 1:
Consider a subset $A\subseteq X$. If $\begin{array}{c} P \quad Q \\ {}_p\searrow \ \swarrow_q \\ X \end{array}$ are principal G-bdles over X, which are both trivial over some open set $U\supset\overline{X-A}$ ("closure"), that retracts to $U\cap A$, Then any isomorphism $\begin{array}{c} P|_A \xrightarrow[\varphi_A]{\cong} Q|_A \\ \searrow \ \swarrow \\ A \end{array}$ extends to an isomorphism $\begin{array}{c} P \xrightarrow[\varphi]{\cong} Q \\ \searrow \ \swarrow \\ X \end{array}$.

Pf: To simplify notation, we'll identify $P|_U$ and $Q|_U$ with $U\times G$.

We extend φ_A using the diagram



So we define

$$\varphi(u,g) = (u, \pi_2(\varphi_A(ru,g)))$$

where $\pi_2 : \underset{\substack{\| \\ Q|_U}}{U \times G} \longrightarrow G$.

This agrees with φ_A on any point in $Q|_A$. To see that φ is continuous, just note that since $U \supset \overline{X-A}$, we have $X = \text{int}(A) \cup U$, and $P = P|_{\text{int}(A)} \cup P|_U$.

So we have defined φ on two open sets, and our def'ns agree on the overlap. $\square$

<u>Lemma 2</u>: For any principal G-bdl $\begin{array}{c} P \\ \downarrow \\ B \times I \end{array}$ and any pt. $b \in B$, $\exists$ an open nbhd $U_b \ni b$ s.t. $P|_{U_b \times I} \cong U_b \times I \times G$.

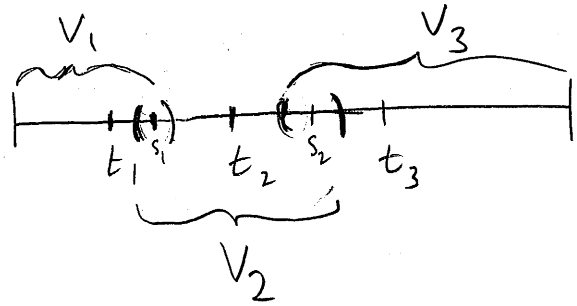
<u>Pf</u>: For each $t \in I$, $\exists$ a nbhd $U_{b,t} \times V_t$ of (b,t) in $B \times I$ over which P is trivial. By cptness of I, $\exists\, t_1, \dots, t_n \in I$ s.t. $I = \bigcup_{i=1}^{n} V_{t_i}$. If we set $U_b = \bigcap_{i=1}^{n} U_{b,t_i}$, then

P is trivial over $U_b \times V_{t_i}$ for each i. WLOG, $t_1 < t_2 < \cdots < t_n$ and no proper subset of $\{V_{t_i}\}_{i=1}^n$ covers I. This implies that $V_{t_i} \cap V_{t_{i+1}} \neq \emptyset$ $(i=1,\dots,n-1)$, and we may choose $s_i \in V_{t_i} \cap V_{t_{i+1}}$. Now set $A_i = [0, s_i]$ and $X_i = \bigcup_{j=1}^{i+1} V_j$. Then

closure in X_i

$\overline{X_i - A_i} \subset X_i \cap [s_i, 1] \subseteq V_{i+1}$ and V_{i+1} retracts to $V_{i+1} \cap A_i$.

The corresponding statements hold after crossing with U_b, so by Lemma 1 and induction on i, we find that P is trivial over $U_b \times X_i$ for each i. $\square$

Picture: 

Note: We've just used the special case of Lemma 1 in which the bundle Q is itself trivial.

Proof of the BHT: Let $\begin{array}{c}P\\ \downarrow\\ B\times I\end{array}$ be a princ. G-bdle.

By Lemma 2, there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of B such that $P|_{U_\alpha\times I}$ is trivial for each $\alpha\in A$.

Since B is paracompact, we may choose a partition of unity $\{\psi_\alpha\}_{\alpha\in A}$ subordinate to $\{U_\alpha\}_\alpha$ (so $\psi_\alpha : B \to [0,1]$, $\mathrm{supp}(\psi_\alpha)\subset U_\alpha$, $\{\alpha : \psi_\alpha(b)\neq 0\}$ is finite for each $b\in B$, and $\sum_{\alpha\in A}\psi_\alpha(b) = 1\ \forall b\in B$).

Choose a well-ordering of A, and set $W_\alpha = \{(b,t) : t \le \sum_{\beta<\alpha}\psi_\beta(b)\}\subset B\times I$.

We want to show $\exists\ \varphi_\alpha : P|_{W_\alpha} \xrightarrow{\cong} (P_0\times I)|_{W_\alpha}$ such that $\varphi_\alpha|_{W_\beta} = \varphi_\beta$ for $\beta<\alpha$. Then $\bigcup_{\alpha\in A}\varphi_\alpha$ is a well-defined function, and its domain $P|_{\bigcup_{\alpha\in A} W_\alpha}$ is in fact all of P, b/c $\sum_{\alpha\in A}\varphi_\alpha \equiv 1 \Rightarrow \bigcup_{\alpha\in A} W_\alpha = B\times I$. So $\bigcup\varphi_\alpha$ will be the desired isomorphism.

To start, note that if $\alpha_0\in A$ is the minimum element, then $W_{\alpha_0} = \{(b,t) : t\le \sum_{\beta<\alpha_0}\psi_\beta(b) = 0\} = B\times\{0\}$, so $P|_{W_{\alpha_0}} = P_0$ by definition of P_0. So an isom. φ_{α_0} exists as desired.

Next, given any $\alpha \in A$, let $\alpha+1 = \min\{\beta \in A : \alpha < \beta\}$. We claim that if $\varphi_\alpha : P|_{W_\alpha} \xrightarrow{\cong} P_0 \times I|_{W_\alpha}$ is an isom., then $\exists\ \varphi_{\alpha+1} : P|_{W_{\alpha+1}} \xrightarrow{\cong} (P_0 \times I)|_{W_{\alpha+1}}$ extending φ_α. We'll prove this using Lemma 1. We have a retraction

$$\begin{aligned} W_{\alpha+1} &\longrightarrow W_\alpha \\ (b,t) &\longmapsto \Big(b, \min\big(t, \sum_{\beta<\alpha} \varphi_\beta(b)\big)\Big) \end{aligned},$$

and $P \times I$, $P_0 \times I$ are both trivial on $(U_\alpha \times I) \cap W_{\alpha+1}$.

To apply Lemma 1, it remains to check that

$$\overline{W_{\alpha+1} - W_\alpha} \subseteq U_\alpha \times I.$$

We have $\overline{W_{\alpha+1} - W_\alpha} \subseteq \operatorname{supp}(\varphi_\alpha) \times I \subseteq U_\alpha \times I$.

Call a subset $A' \subset A$ <u>closed</u> if whenever $\alpha \in A$ and $\alpha < \alpha'$ for some $\alpha' \in A'$, then in fact $\alpha \in A'$.

Consider the partially ordered set

$$\mathcal{P} = \Big\{ \{\varphi_\alpha\}_{\alpha \in A'} : A' \subset A \text{ is closed}, \ \varphi_\alpha : P|_{W_\alpha} \xrightarrow{\cong} P_0 \times I|_{W_\alpha} \text{ for each } \alpha \in A', \text{ and } \varphi_\alpha|_{W_\beta} = \varphi_\beta \text{ whenever } \beta < \alpha \ (\beta \in A') \Big\}.$$

Here $\mathcal{P}$ is ordered by inclusion, i.e. $\{\varphi_\alpha\}_{\alpha \in A'} < \{\varphi'_\alpha\}_{\alpha \in A''}$ if $A' \subseteq A''$ and $\varphi_\alpha = \varphi'_\alpha \ \forall \alpha \in A'$.

If C is a chain in $\mathcal{P}$, then we claim that the union of C (that is, the union of all the sets of functions appearing in C) is still in $\mathcal{P}$, hence is an upper bound for C. The only thing to check is that the union of a chain of closed subsets of A is still closed, but this is straightforward.

Now by Zorn's Lemma, $\mathcal{P}$ has a maximum element. We claim that this max. elt. must be of the form $\{\varphi_\alpha\}_{\alpha\in A'}$ with $A'=A$, which will complete the proof. If not, then $\exists\ \alpha \in A\setminus A'$, and since A' is closed, we have $\alpha > \alpha'$ for all $\alpha' \in A'$. Let $\alpha_0 = \min(A\setminus A')$, so that $A' = \{\alpha' \in A : \alpha' < \alpha_0\}$. If A' has a maximum element $m \in A'$, then $\alpha_0 = m+1$, and our previous argument shows that φ_m can be extended to $\varphi_{\alpha_0} : P|_{W_{\alpha_0}} \xrightarrow{\sim} P_0 \times I|_{W_{\alpha_0}}$, contradicting maximality of A'. If A' does not have a max. elt, then

$W_{\alpha_0} = \{(b,t) : t \leq \sum_{\beta < \alpha_0} \varphi_\beta(t)\} = \bigcup_{\alpha' < \alpha_0} W_{\alpha'}$,

so $\{\varphi_{\alpha'}\}_{\alpha' \in A'}$ already defines an isom.

$P|_{W_{\alpha_0}} \xrightarrow{\cong} P_0 \times I|_{W_{\alpha_0}}$. Again, this contradicts

maximality of $\{\varphi_\alpha\}_{\alpha \in A'}$. $\square$

<u>Characteristic Classes</u>:

<u>Defn</u>: A <u>characteristic class</u> for principal G-bundles is a natural transformation

$$\mathrm{Prin}_G(X) \xrightarrow{c} H^n(X;A),$$

defined for all CW complexes X. (n and A are fixed; the are the dim'n and coeff. gp for the class).

Naturality just means that given a map $f: X \to Y$ and a bdle $\begin{smallmatrix} P \\ \downarrow \\ Y \end{smallmatrix}$, we have $c(f^*Y) = f^*(c(P)) \in H^n(X;A)$.

<u>Lemma (Yoneda)</u>:

Say $\begin{smallmatrix} EG \\ \downarrow \\ BG \end{smallmatrix}$ is a universal principal G-bdle w/ B a CW cplx. Then there is a bijection

$$\left\{ \begin{matrix} \text{Char. classes for } G\text{-bdles} \\ \text{w/ coeff. gp. } A \text{ and dim'n } n \end{matrix} \right\} \xrightarrow{\cong} H^n(BG;A).$$

<u>Pf</u>: The bijection is given by $c \longmapsto c(EG) \in H^n(BG;A)$. It remains only to note that for any $\alpha \in H^n(BG;A)$, the assignment $P \longmapsto f^*(\alpha)$ (where f is a classifying map for P) defines a characteristic class; this uses the bijection $\mathrm{Prin}_G(X) \cong [X, BG]$. $\square$