Math 697 - Homework 4
Due Wednesday, May 3

**Problem 1 (The multiplication on $\mathbb{C}P^\infty$):** In class we studied the "multiplication" map $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ (we defined $\mu$ to be a classifying map for the line bundle $\pi_1^*\gamma^1 \otimes \pi_2^*\gamma^1 \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$). Prove that $\mu$ is a homotopy commutative and has an inverse up to homotopy. That is, show that

$$\mu \circ \tau \simeq \mu,$$

where

$$\tau : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$$

is the "twist" map $\tau(x, y) = (y, x)$, and show that there is a map $\eta\colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ such that

$$\mu \circ (\eta \times \mathrm{Id}) \circ \Delta \simeq \mu \circ (\mathrm{Id} \times \eta) \circ \Delta \simeq c,$$

where Id is the identity map on $\mathbb{C}P^\infty$, $\Delta\colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ is the diagonal $x \mapsto (x, x)$, and $c$ is the constant map $\mathbb{C}P^\infty \to * \in \mathbb{C}P^\infty$ (the choice of $*$ is irrelevant since we're working up to homotopy). (Hint: these are similar to the proof that $\mu$ has an identity up to homotopy, which was proven in class. For the inverse, think about dual bundles.)

**Remark:** The multiplication $\mu$ is also homotopy associative, meaning that $\mu \circ (\mathrm{Id} \times \mu) \simeq \mu \circ (\mu \times \mathrm{Id})$, and the proof is similar.

**Problem 2 (Determinants and the first Chern class):**

Let $E \to B$ be a vector bundle (real or complex). Let $\det(E) \to B$ be the line bundle formed by applying the continuous functor sending an $n$-dimensional vector space $V$ to its top exterior power $\Lambda^n V$ (and sending an isomorphism $V \to W$ to the induced map $\Lambda^n V \to \Lambda^n W$).

a) Consider the case in which $B = U_1 \cup U_2$, and $E$ is trivial over both $U_1$ and $U_2$. Show that if the transition function for $E$ is

$$\phi\colon\ U_1 \cap U_2 \to \mathrm{GL}(n),$$

then the transition function for $\det(E)$ is

$$\det \circ \phi\colon\ U_1 \cap U_2 \to \mathrm{GL}(1),$$

where det is the usual determinant map.

b) Prove that for any complex vector bundle $E$ (over a paracompact base space), we have $c_1(E) = c_1(\det E)$. (Hint: use the Splitting Principle; that is, begin with the case where $E$ is a direct sum of line bundles. Paracompactness is just needed to ensure that the Splitting Principle applies.)

**Problem 3 (Characteristic classes of tensor products):** Given 2-dimensional complex vector bundles $E$ and $F$ over a space $X$, compute the characteristic classes $c_1(E \otimes F)$ and $c_2(E \otimes F)$ in terms of the classes $c_1(E), c_2(E), c_1(F)$, and $c_2(F)$. (Hint: use the Splitting Principle.)

**Remark:** It's not much harder to compute $c_3(E \otimes F)$ and $c_4(E \otimes F)$, but the formulas get rather large.