Math 697 - Characteristic Classes
Homework 3 - Due Wednesday, April 5

General hint: the Whitney Sum Formula is needed for most of these problems.

Problem 1: Stable isomorphisms
a) Two real vector bundles $E \to B$, $F \to B$ are said to be *stably isomorphic* if there exists n such that $E \oplus \epsilon^n \cong F \oplus \epsilon^n$, where $\epsilon^n = B \times \mathbb{R}^n$ is the trivial n–plane bundle over B.

Prove that if E and F are stably isomorphic, then $w_i(E) = w_i(F)$ for each $i = 0, 1, \ldots$. (The analogous result holds in the complex case, with the same proof.)

b) Prove that $w_i(TS^n) = 0$ for all i, n. (Note: the tangent bundle of S^n is *not* always trivial! For instance, there is no everywhere-nonzero vector field on S^2.)

c) Prove that $w_i(TM^g) = 0$ for all i, g, where M^g is the closed orientable surface of genus g. (Again, the tangent bundle is not trivial.)

Problem 2: The Splitting Principle
a) Prove the following alternate form of the Splitting Principle: For each $n \geqslant 1$, the classifying map

$$(\mathrm{Gr}_1(\mathbb{C}^\infty))^n \xrightarrow{\alpha} \mathrm{Gr}_n(\mathbb{C}^\infty)$$

for the complex n–plane bundle

$$\gamma_1 \times \cdots \times \gamma_1 \longrightarrow \mathrm{Gr}_1(\mathbb{C}^\infty) \times \cdots \times \mathrm{Gr}_1(\mathbb{C}^\infty)$$

induces an injection on integral cohomology. Here $\gamma_1 \to \mathrm{Gr}_1(\mathbb{C}^\infty)$ is the tautological line bundle over $\mathrm{Gr}_1(\mathbb{C}^\infty) \cong \mathbb{C}P^\infty$. (Hint: use the Projective Bundle Theorem.)

b) By the Künneth Theorem, $H^*((\mathrm{Gr}_1(\mathbb{C}^\infty))^n; \mathbb{Z})$ is the polynomial ring on 2-dimensional the classes $x_i := \pi_i^*(c_1(\gamma_1))$, where π_i is projection to the ith factor (recall here that, by definition, $c_1(\gamma_1) \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$ is a generator).

Prove that $\alpha^*(c_i(\gamma_n))$ is the ith elementary symmetric polynomial in the x_i.

Remark: This problem also shows that the Chern classes of γ_n are never zero, since they pull back to non-zero classes in the polynomial ring $H^*((\mathrm{Gr}_1(\mathbb{C}^\infty))^n; \mathbb{Z})$. The same argument shows the Stiefel–Whitney classes of the real universal bundles are non-zero.

In fact, more is true. It's a basic algebraic fact that there are no polynomial relations amongst the elementary symmetric polynomials. This means that the classes $c_i(\gamma_n)$ are algebraically independent in $H^*(\mathrm{Gr}_n(\mathbb{C}^\infty); \mathbb{Z})$. In fact, these classes generate the cohomology ring $H^*(\mathrm{Gr}_n(\mathbb{C}^\infty); \mathbb{Z})$ (see Milnor–Stasheff §14), i.e. this ring is the polynomial algebra on the Chern classes of the universal bundle γ_n. Again, the analogous statements hold in the real case.

Problem 3:
a) Prove that the universal bundle $\gamma_n \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ is *not* orientable, but

$$\gamma_n \oplus \gamma_n \to \mathrm{Gr}_n(\mathbb{R}^\infty)$$

is orientable.

b) Prove that $w_{2n}(\gamma_n \oplus \gamma_n) \neq 0$. (Hint: Problem 2 is helpful here.)

Problem 4: Dual Bundles
In this problem, $\mathbb{F}$ will denote either the field $\mathbb{R}$ of real number, or the field $\mathbb{C}$ of complex numbers. Given a vector bundle $E \to B$ with fiber $\mathbb{F}^n$, the dual bundle $E^* = \mathrm{Hom}_F(E, \mathbb{F})$ is defined using the construction in Milnor–Stasheff Chapter 3.

Remark: There is a subtlety here: You need to express the dual space as a *covariant functor* from vector spaces over $\mathbb{F}$ (and isomorphisms) to vector spaces over $\mathbb{F}$ (and isomorphisms), so that the construction in Milnor–Stasheff applies. This just amounts to inverting the usual map on dual spaces induced by a linear map.

a) Say $\phi_i : U_i \times \mathbb{F}^n \to E|_{U_i}$ are trivializations with $U_1 \cap U_2 \neq \emptyset$. Describe the transition function for E^* in terms of the transition function $\phi_2^{-1}\phi_1$. (Hint: it's important to notice that in Milnor–Stasheff the trivializations of E^* come from $(\mathbb{F}^n)^*$, rather than from F^n, so you'll need to compose with the natural isomorphism between $(\mathbb{F}^n)^*$ and $\mathbb{F}^n$.)

b) Consider the natural embedding $\mathbb{F}P^1 \to \mathbb{F}P^2$ given by sending $[x, y] \in \mathbb{F}P^1 = (\mathbb{F}^2 - \{0\})/\mathbb{F}^\times$ to $[x, y, 0] \in \mathbb{F}P^2 = (\mathbb{F}^3 - \{0\})/\mathbb{F}^\times$. There is a natural projection

$$\pi : \mathbb{F}P^2 - \{[0, 0, 1]\} \longrightarrow \mathbb{F}P^1$$

given by $[x, y, z] \mapsto [x, y, 0]$ (where we identify $\mathbb{F}P^1$ with its image under the embedding), and each fiber of this projection is just a copy of $\mathbb{F}$. Show that this projection is locally trivial over the open sets $\{[x, y, 0] \in \mathbb{F}P^1 : x \neq 0\}$ and $\{[x, y, 0] \in \mathbb{F}P^1 : y \neq 0\}$ (hence π is a vector bundle), and compute the transition function defined by the two trivializations. (This bundle over $\mathbb{F}P^1$ is the *normal bundle* of $\mathbb{F}P^1$ in $\mathbb{F}P^2$.)

c) Recall that the tautological bundle over $\mathbb{F}P^1$ can be written as

$$\{[x, y], \vec{\mathbf{v}} \in \mathbb{F}P^1 \times \mathbb{F}^2 : \vec{\mathbf{v}} = (tx, ty) \text{ for some } t \in \mathbb{F}\}$$

Prove that when $\mathbb{F} = \mathbb{R}$, the normal bundle from b) is isomorphic to the tautological bundle γ_1^1 over $\mathbb{RP}^1 \cong S^1$ (the "Mobius" bundle), and that when $\mathbb{F} = \mathbb{C}$, the normal bundle is the *dual* of the tautological bundle over $\mathbb{C}P^1 \cong S^2$. (Hint: compare transition functions.)

Remark: In the real case, any metric on a bundle E determines an isomorphism $E \cong E^*$. In the complex case, this is *not* true, because a Hermitian metric is *conjugate linear* in the second coordinate. In the complex case, one gets an isomorphism $E^* \cong \overline{E}$, where $\overline{E}$ is the bundle with the same underlying total space as E, but with the opposite complex structure: in the bundle $\overline{E}$, multiplication by $z \in \mathbb{C}$ acts the way multiplication by $\overline{z}$ acts on E. The bundles E and $E^* \cong \overline{E}$ are not isomorphic in general, and in particular $c_1(\overline{E}) = -c_1(E)$. See MS Chapter 14.