Math 697 - Characteristic Classes
Homework 2 - Due Wednesday, March 8

Problem 1: Homotopy groups of classifying spaces

a) Recall that given a group G, the universal principal G–bundle is a principal G–bundle

$$EG \longrightarrow BG$$

for which $\pi_*(EG) = 0$ for $* = 0, 1, \ldots$. Use the long exact sequence in homotopy associated to the fibration $EG \to BG$ to describe the relationship between $\pi_* BG$ and $\pi_* G$.

In the rest of the problem, you may use the following result.

Theorem The Stiefel manifold $V_n(\mathbb{C}^\infty)$ of n–frames in $\mathbb{C}^\infty$ is a universal principal $\mathrm{GL}_n(\mathbb{C})$–bundle over $\mathrm{Gr}_n(\mathbb{C}^\infty)$, and the submanifold $V_n^0(\mathbb{C}^\infty)$ of unitary n–frames is a principal $U(n)$–bundle over $\mathrm{Gr}_n(\mathbb{C}^\infty)$.

The proof of this Theorem is a straightforward modification of the arguments given in class in the real case. To compute homotopy groups, one again uses the natural projections $V_n^0(\mathbb{C}^{n+k}) \to V_{n-1}^0(\mathbb{C}^{n+k})$, which are fiber bundles with fiber the unit sphere in $\mathbb{C}^k$, i.e. S^{2k-1}. From the long exact sequence in homotopy, one finds that $\pi_*(V_n^0(\mathbb{C}^{n+k})) = 0$ for $* < 2k - 1$.

b) The action of the unitary group $U(n)$ on $\mathbb{C}^n$ preserves lengths, and hence $U(n)$ acts on $S^{2n-1} \subset \mathbb{C}^n$. Show that this action is transitive, and that the stabilizer of a point under this action is homeomorphic $U(n-1)$. This gives a homeomorphism $U(n)/U(n-1) \cong S^{2n-1}$. Show that the quotient map

$$U(n) \longrightarrow U(n)/U(n-1) \cong S^{2n-1}$$

is in fact a fiber bundle with fiber $U(n-1)$.

c) Show that $U(n)$ is path-connected, and compute $\pi_i U(n)$ for $i = 1, 2, 3$ (you may use the facts that $\pi_k(S^k) \cong \mathbb{Z}$ and $\pi_m(S^k) = 0$ for $m < k$. Then compute $\pi_i \mathrm{Gr}_n(\mathbb{C}^\infty)$ for $i = 0, \ldots, 4$. What can you deduce about complex vector bundles (or, equivalently, principal $\mathrm{GL}_n(\mathbb{C})$–bundles) over S^1, S^2, S^3, and S^4?

Problem 2: The Gauss map

Let $M \subset \mathbb{R}^n$ be a d–dimensional smooth manifold, in the sense defined in Chapter 1 of Milnor–Stasheff. The *Gauss map* $\beta \colon M \to \mathrm{Gr}_d(\mathbb{R}^n)$ is defined by sending a point $x \in M$ to the tangent plane $T_x M$, considered as a point in $\mathrm{Gr}_d(\mathbb{R}^n)$.

Prove that the Gauss map is a continuous map that classifies the tangent bundle of M. In other words, show that the pullback $\beta^*(\gamma^d(\mathbb{R}^n)) \to M$ is isomorphic to TM, where

$$\gamma^d(\mathbb{R}^n) = \{(P, \vec{v}) \colon P \in \mathrm{Gr}_d(\mathbb{R}^n), \vec{v} \in P\}$$

is the canonical vector bundle over the Grassmannian.

Problem 3: The Grassmannian as a Homogeneous Space
a) Prove that the orthogonal group $O(n+k)$ acts transitively on $\mathrm{Gr}_n(\mathbb{R}^{n+k})$. Identify the stabilizer $H \leqslant O(n+k)$ of the n–plane $\mathbb{R}^n \oplus 0$ under this action, and show that $\mathrm{Gr}_n(\mathbb{R}^{n+k}) \cong O(n+k)/H$

b) Prove that the unitary group $U(n+k)$ acts transitively on $\mathrm{Gr}_n(\mathbb{C}^{n+k})$. Identify the stabilizer $K \leqslant U(n+k)$ of the n–plane $\mathbb{C}^n \oplus 0$ under this action, and show that $\mathrm{Gr}_n(\mathbb{C}^{n+k}) \cong U(n+k)/K$

Problem 4: Orientations
An orientation of a finite-dimensional real vector space V is an equivalence class of ordered bases, under the equivalence relation $(\vec{v}_1, \ldots, \vec{v}_n) \sim (\vec{w}_1, \ldots, \vec{w}_n)$ if the linear transformation $\vec{v}_i \mapsto \vec{w}_i$ has positive determinant. (So each n-dimensional vector space, for $n \geqslant 1$, has exactly 2 orientations.) An n–dimensional vector bundle $E \to B$ is orientable if there exists a choice of orientation for each fiber of E such that locally the orientations agree: precisely, this means that for each $x \in B$, there exists a local trivialization $E|_U \xrightarrow{\cong} U \times \mathbb{R}^n$ which maps the chosen orientations to the standard orientation $[e_1, \ldots, e_n]$ for $\mathbb{R}^n$.

Let $\mathrm{Gr}_n^{or}(\mathbb{R}^{n+k})$ be the quotient of the Steifel manifold $V_n(\mathbb{R}^{n+k})$ by the action of $\mathrm{GL}_n^+(\mathbb{R})$ (the group of positive determinant matrices).

a) Prove that the quotient map

$$\pi\colon\ \mathrm{Gr}_n^{or}(\mathbb{R}^{n+k}) = V_n(\mathbb{R}^{n+k})/\mathrm{GL}_n^+(\mathbb{R}) \longrightarrow \mathrm{Gr}_n(\mathbb{R}^{n+k}) = V_n(\mathbb{R}^{n+k})/\mathrm{GL}_n(\mathbb{R})$$

is a 2-sheeted covering space.

b) Prove that the pullback

$$\pi^*(\gamma^n(\mathbb{R}^{n+k})) \longrightarrow \mathrm{Gr}_n^{or}(\mathbb{R}^{n+k})$$

is an orientable vector bundle. (Hint: the open neighborhoods $\mathcal{O}_F$ discussed in class are simply connected - in fact they are homeomorphic to $\mathbb{R}^{n+k}$; see the proof of Lemma 5.1 in Milnor–Stasheff.)

c) Let B be a CW complex. We have shown in class that each vector bundle $E \to B$ is classified by a map (unique up to homotopy) $B \to \mathrm{Gr}_n(\mathbb{R}^{n+k})$ for large enough k (see also Lemma 5.3 and Corollary 5.10 in Milnor–Stasheff).

Prove that a vector bundle $E \to B$ is orientable if and only if its classifying map $f\colon\ B \to \mathrm{Gr}_n(\mathbb{R}^{n+k})$ lifts to $\mathrm{Gr}_n^+(\mathbb{R}^{n+k})$, in the sense that there exists a map $\widetilde{f}\colon\ B \to \mathrm{Gr}_n^+(\mathbb{R}^{n+k})$ such that $\pi \circ \widetilde{f} = f$, where π is the covering map from part a).

d) Using part B), prove that a vector bundle $E \to B$, with B a CW complex, is orientable if and only if for every loop $\gamma\colon\ S^1 \to B$, the pullback $\gamma^*(E) \to S^1$ is orientable. (Hint: use some covering space theory.)

Remark The last condition is, by definition, the same as saying that the 1st Stiefel-Whitney class $w_1(E)$ is trivial. This is because a vector bundle over S^1 is orientable if and only if it's trivial (there are only two vector bundles over S^1, the Mobius bundle and the trivial bundle).