Math 697 - Characteristic Classes Homework I

Due by noon on Friday, February 3.

Problem 1 (Smooth maps)

Let $M^d \subset \mathbb{R}^n$ be a d–manifold in the sense defined in class (or in Milnor–Stasheff), and let $V \subset \mathbb{R}^m$ be an open subset of some Euclidean space. Show that the following two conditions on a function $f : V \to M^d$ are equivalent. (Functions satisfying these two equivalent conditions are called *smooth*.)

- The composition $i \circ f \colon \mathbb{R}^m \to \mathbb{R}^n$ is smooth, where $i : M^d \hookrightarrow \mathbb{R}^n$ is the inclusion.
- For each chart $h \colon U \to M^d$, the composition $h^{-1}f \colon V \to U$ is smooth.

Hint: For the harder direction, you'll need to use the Inverse Function Theorem, which states that if $f : W_1 \to W_2$ is a smooth function between open sets in $\mathbb{R}^n$, and its Jacobian $D_x f$ is invertible at some point $x \in W_1$, then there are open sets $W_1' \subset W_1$ containing x and $W_2' \subset W_2$ containing $f(x)$, and a smooth map $g : W_2' \to W_1'$ such that g is inverse to (the restriction of) f.

Remark: This problem gives a well-defined notion of smooth maps $f : M^d \to N^k$ between manifolds: for each chart $h : U \to M^d$, we require that $f \circ h : U \to N^k$ is smooth in the above sense.

Problem 2 (Projective spaces, see MS Problem 1-B part b)

A) Given a vector $\overrightarrow{\mathbf{x}} = (x_1, \ldots, x_{k+1}) \in S^k$, let $\alpha(\overrightarrow{\mathbf{x}})$ be the $(k+1) \times (k+1)$–matrix with (i,j)th entry $x_i x_j$. Show that the mapping

$$\alpha : S^k \longrightarrow M_{k+1}(\mathbb{R})$$

induces a homeomorphism from $\mathbb{R}P^k$ to the subspace

$$P^k = \{A \in M_{k+1}(\mathbb{R}) \mid A^T = A,\ AA = A,\ \text{ and trace}(A) = 1\}.$$

B) Show that the subspace $P^k \subset M_{k+1}(\mathbb{R}) \cong \mathbb{R}^{(k+1)^2}$ is a smooth manifold of dimension k. (This now gives $\mathbb{R}P^k$ the structure of a smooth manifold.)

C) The tangent bundle to S^k has an action of the group $\mathbb{Z}/2$, given by sending $(x, \alpha'(0))$ to $(-x, -\alpha'(0))$ (where $x \in S^k$ and α is a smooth curve in S^k with $\alpha(0) = x$). Show that α is a smooth map, and that $D\alpha : TS^k \to TP^k$ induces a homeomorphism from $TS^k/(\mathbb{Z}/2)$ to TP^k.

Problem 3 (Principal Bundles)

A) Let X be a topological space and G a topological group with identity element $e \in G$. Say G acts continuously on X (that is, the map $X \times G \to X$ is continuous). Show that the quotient map $X \xrightarrow{q} X/G$ has the structure of a principal G–bundle if and only if the following three conditions are satisfied:

(1) The action is free, meaning that if $x \cdot g = x$ for some $x \in X$ and some $g \in G$, then $g = e$.

(2) There exists an open cover $\{U_i\}_{i \in I}$ of X/G and continuous functions $s_i : U_i \to X$ such that $q \circ s_i = \mathrm{Id}$ (we say that the s_i are *local sections*, or *slices*).

(3) The translation map $t : X \times_{X/G} X \to G$, defined by setting $t(x, y)$ to be the unique $g \in G$ such that $g \cdot x = y$, is a continuous map.

B) Given an n–dimensional vector bundle $p\colon E \to B$, let

$$\mathrm{Fr}(E) = \{(v_1, \ldots, v_n) \in E^n \,:\, p(v_1) = \cdots = p(v_n) \text{ and } (v_1, \ldots, v_n) \text{ is linearly independent}\}$$

and let $\pi\colon \mathrm{Fr}(E) \to B$ be the map $\pi(v_1, \ldots, v_n) = p(v_1)$. Show there is an action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathrm{Fr}(E)$ such that $\mathrm{Fr}(E)/\mathrm{GL}_n(\mathbb{R})$ is homeomorphic to B.

C) Using parts A) and B), show that the for any vector bundle $E \to B$, the frame bundle $\mathrm{Fr}(E) \to B$ is a principal $\mathrm{GL}_n(\mathbb{R})$–bundle. (This is a little bit confusing, because strictly speaking, A) refers to the quotient map $\mathrm{Fr}(E) \to \mathrm{Fr}(E)/\mathrm{GL}_n(\mathbb{R})$. But by B), this quotient space is homeomorphic to B.) Prove that $\mathrm{Fr}(E) \to B$ is isomorphic to the principal $\mathrm{GL}_n(\mathbb{R})$–bundle obtained from E via clutching.

D) Modify the ideas in A) - C) to show that if $E \to B$ is a Euclidean bundle, then the subspace of $\mathrm{Fr}(E)$ consisting of *orthonormal* frames is a principal $O(n)$–bundle over B, isomorphic to the bundle obtained via clutching.

Problem 4 (Mixing)

A) Show that if $P \to B$ is a principal $O(n)$–bundle, then the mixed bundle $P \times_{O(n)} \mathbb{R}^n \to B$ is a Euclidean vector bundle (in other words, show that there is a natural way to put a metric on this bundle).

B) Prove that the operations of forming frame bundles and mixing are inverse to one another (up to isomorphism). In other words, if $P \to B$ is a principal $GL_n(\mathbb{R})$–bundle and $E \to B$ is a vector bundle, then

$$\mathrm{Fr}\left(P \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n\right) \cong P \quad \text{and} \quad (\mathrm{Fr}(E)) \times_{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}^n \cong E,$$

and similarly in the Euclidean case (replacing $\mathrm{GL}_n(\mathbb{R})$ by $O(n)$).

Problem 5 (Metrics)

A) (MS Problem 2-C) Using a partition of unity, show that every vector bundle over a paracompact space can be given a Euclidean metric.

B) Use the Bundle Homotopy Theorem to show that if $\langle\, , \rangle$ and $\langle\, , \rangle'$ are two metrics on the same vector bundle $E \to B$, then there is a bundle isomorphism $\phi : E \to E$ such that $\langle \phi(\vec{\mathbf{v}}), \phi(\vec{\mathbf{w}})\rangle' = \langle \vec{\mathbf{v}}, \vec{\mathbf{w}} \rangle$ for all $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in E$.

C) EXTRA CREDIT: Give a direct construction of a map ϕ satisfying part b). See MS Problem 2-E for a description of how to do this.