Oriented Bdles + the Euler Class (MS §9)

The Euler class is an *integral* char. class associated to *real* v. bdles: for any v. bdle $\mathbb{R}^n \to E \to X$, the Euler class is an element $e(E) \in H^n(X;\mathbb{Z})$, defined when E is "orientable".

Theorem (MS 11.12)
Let M^n be an orientable closed (smooth) mfld. Then the integer

$$\langle e(TM^n), [M] \rangle = \chi(M),$$

the Euler Characteristic of M (i.e. $\chi(M) = \sum_{i=0}^{n} (-1)^i \,\overset{\text{Rank}}{Rk}\, H_i(M;\mathbb{Z})$).

(In particular, if M^n is orientable, then TM^n is an orientable real bdle)

Theorem: (MS 9.7)
If an orientable closed mfld (smooth) admits a nowhere vanishing tangent vector field then $\chi(M) = 0$.

Milnor proves this by showing that the Euler class is the "primary obstruction" to the existence of a nowhere zero section of $E \to X$ defined on the n-skeleton $X^{(n)}$! So if M^n has a nowhere-zero vector field, i.e. a nowhere-zero section of TM^n, then this primary obstruction must vanish, and then $\chi(M) = \langle e(TM^n), [M] \rangle = 0$.

<u>Example</u>: $\chi(S^{2n}) = 2 \neq 0$, so S^{2n} does not admit a nowhere vanishing vector field.

To begin, we need to discuss <u>orientations</u>.

<u>Def'n</u>: An <u>orientation</u> of a real v. space V is an equivalence class of ordered bases, under the equiv. rel'n

$[v_1, \dots, v_n] \sim [w_1, \dots, w_n]$ if the transformation $v_i \mapsto w_i$ has determinant greater than zero.

<u>Def'n</u> A vector bdle $\begin{matrix} \mathbb{R}^n \to E \\ \downarrow \\ X \end{matrix}$ is orientable if one of the following equivalent conditions is satisfied:

1) E admits a (continuous) orientation; that is, there exist orientations for each fiber and trivializations $E|_u \xrightarrow{\cong} u \times \mathbb{R}^n$ which send the chosen orientations to the standard orientation $[e_1, \dots, e_n]$ of $\mathbb{R}^n$.

2) E admits transition fcns w/ positive det's (i.e. there is a trivialization in which all φ_{ij} have $\det \varphi_{ij} > 0$)

3) E is the mixed bdle associated to a principal $GL_n^+\mathbb{R}$-bdle, where $GL_n^+\mathbb{R} = \{A \in GL_n\mathbb{R} \mid \det A > 0\}$.

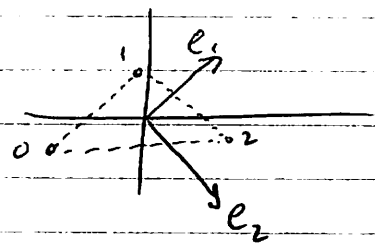
4) E is the mixed bdle associated to a principal $SO(n)$-bdle.

Pf (1) and 2) are equiv. essentially by defn; 2) and 3) are exactly the same by our prior discussion of mixed bdles / assoc. principal bdles, and 4) arises via Gram-Schmidt and various HW problems.)

Orientations on a real vector space V determine generators of the group $H_n(V, V_0; \mathbb{Z})$, where $V_0 = V - \{0\}$:

$H_n(D^n, S^{n-1}; \mathbb{Z}) \cong H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$

- Given an orientation $[e_1, \dots, e_n]$, choose a singular n-simplex $\Delta^n \to \mathbb{R}^n$ whose interior contains 0 and s.t. the vector pointing from its i^{th} vertex to its $(i+1)^{st}$ vertex is e_i:



Then this singular simplex lies in the relative cycles $Z_n(V, V_0)$, and maps to a generator of $H_n(S^n; \mathbb{Z})$.

There is then a dual generator of $H^n(V, V_0; \mathbb{Z}) \cong H^n(S^n; \mathbb{Z})$ determined by the orientation.

More generally, if $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is an oriented v. bdle (i.e. we have <u>specified</u> an orientation on E) then <u>locally</u> we can choose consistent generators of $H_n(E_x, (E_x)_0; \mathbb{Z})$: locally $E|_U \cong U \times \mathbb{R}^n$, and (with the right choose of trivializations) the orientation of E_x maps to the standard orientation on $\mathbb{R}^n$. The cohomology gp

$H^n(U \times \mathbb{R}^n, U \times \mathbb{R}^n_0; \mathbb{Z}) \cong H^n(U \times S^n; \mathbb{Z})$ now has a ~~[illegible]~~ generator restricting to the orientation generator in each fiber, and we can transfer this generator back to $E|_U$.

Hence associated to the orientation, we have a covering $\{U_i\}$ of B and classes in $H^n(E|_{U}, E|_{U})_0; \mathbb{Z})$ which restrict in each fiber to the orientation generators. These classes can actually be glued together over all of E:

<u>Theorem</u> (The Thom Isom Theorem):

If E is an oriented real n-plane bdle, then $H^i(E, E_0; \mathbb{Z}) = 0$ for $i<n$, and $\exists$ a <u>unique</u> class $U \in H^n(E, E_0; \mathbb{Z})$ such that for every $x \in B$, the restriction

"Thom class"

$$u_x \in H^n(E_x, (E_x)_0; \mathbb{Z})$$

is the orientation generator. Moreover, the map

$$H^k(B;\mathbb{Z}) \cong H^k(E;\mathbb{Z}) \longrightarrow H^{n+k}(E, E_0; \mathbb{Z})$$
$$\alpha \longmapsto \alpha \cup U$$

relative cup product

is an isomorphism.

(Note here that $H^k B \cong H^k E$ b/c $\overset{E}{\underset{B}{\downarrow \pi}}$ is a fib'n with fiber $\mathbb{R}^n \simeq *$; hence π ~~is~~ a weak equivalence and induces isom's in H_* and H^*.)

There is now a restriction map $H^*(E, E_0) \to H^*(E, \emptyset) = H^*E$, giving rise to the <u>Euler class</u>:

<u>Def'n</u>: If $\overset{E}{\underset{B}{\downarrow \pi}}$ is an oriented n-plane bdle, its Euler class is $e(E) = (\pi^*)^{-1}(U|_E) \in H^n(B;\mathbb{Z})$.

Thm (Property 9.5) The mod-2 reduction of the Euler class is precisely $w_n(E)$. That is, the map $H^n(B;\mathbb{Z}) \to H^n(B;\mathbb{Z}/2)$ sends $e(E)$ to $w_n(E)$.

With our def'n of the Stiefel-Whitney classes, this is not so clear, unfortunately. One can, however, define all the Stiefel-Whitney classes in terms of the Thom class, and then check that they satisfy the axioms (hence must agree w/ our def'n).

This requires the theory of "cohomology operations" and more specifically Steenrod squares.

Basic Properties of the Euler Class:

1) Naturality: If
$$\begin{array}{ccc} E & \to & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B' \end{array}$$
is an orientation preserving map of bdles, then $f^*(e(E')) = e(E)$.

(This follows from uniqueness of $u(E)$: in each fiber, the orientation generators agree under f^*, so $f^*(u(E'))$ has the defining property of $u(E)$.)

2) Changing the orientation causes $e(E)$ to simply change signs.

(Same reason as above)

3) $2 \cdot e(E) = 0$ if E is an odd-dim'l bdle.

(In odd dim'ns, $\begin{array}{c} \mathbb{R}^n \to \mathbb{R}^n \\ x \longmapsto -x \end{array}$ has negative det., hence is orientation reversing. Now for any bdle E,
$$\begin{array}{ccc} (E,-) & \xrightarrow{-1} & (E,+) \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$
$$\underset{2)}{-e_+(E) = e_-(E)} \underset{1)}{\longleftarrow\!\!\mid} e_+(E)$$
But the map is $B \xrightarrow{id} B$, so $-e_+(E) = e_+(E)$.)

Multiplicativity of the Euler Class:

Note that the Whitney sum formula says that if E is n-dim'l, F is m-dim'l, $w_{2n}(E \oplus F) = w_n E \cup w_m F$.

So after reducing mod 2, Euler classes are multiplicative. In fact:

Theorem (MS 9.6): $e(E_1 \oplus E_2) = e(E_1) \cup e(E_2)$, and $e(E_1 \times E_2) = e(E_1) \times e(E_2)$.

Pf: This is essentially a consequence of the Kunneth Theorem. One finds that the orientation generators are multiplicative, and the rest follows by naturality. $\square$

(This can be used to show that the versions of Stiefel-Whitney classes defined via the Thom class satisfy the Whitney-Sum Formula.)

The Euler Class as an Obstruction

Thm: If $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ is an orientable bdle w/ a nowhere-zero section, then $e(E) = 0$.

Pf: (Assume B is paracpt.) Choosing a metric on B, the span of the section and its orthogonal complement yield a splitting $E \cong E' \oplus \varepsilon'$. Then $e(E) = e(E') \cup e(\varepsilon')$, but $e(\varepsilon') = 0$ (b/c ε' can be pulled back from a point).

Lecture 22

The Thom Isomorphism Theorem

If $\begin{array}{c}\xi\\ \downarrow\\ B\end{array}$ is an oriented (real) n-plane bdle, then

$\exists$ a unique class $u \in H^n(\xi, \xi_0; \mathbb{Z})$ which restricts to

the orientation generator in $H^n(\xi_x, (\xi_x)_0; \mathbb{Z})$ for every $x \in B$.

There is then an isom $H^k(B;\mathbb{Z}) \cong H^k(E;\mathbb{Z}) \xrightarrow[\cong]{\cup u} H^{n+k}(E, E_0; \mathbb{Z})$

for $k \geq 0$, and $H^*(E, E_0; \mathbb{Z}) = 0$ for $* < n$.

Proof: We'll assume B is compact, or at least that

there is a finite open covering of B, $\{U_i\}$, s.t. $\xi|_{U_i} \overset{\text{or. preserving}}{\cong} U_i \times \mathbb{R}^n$.

We have already observed that the Kunneth Thm implies

that there is a unique class $u \in H^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0; \mathbb{Z})$

restricting to the orientation gen's, and moreover the Kunneth isom.

$$\begin{array}{ccccc} H^*(B;\mathbb{Z}) & \otimes & H^n(\mathbb{R}^n, \mathbb{R}^n_0; \mathbb{Z}) & \xrightarrow{\cong} & H^{n+*}(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0; \mathbb{Z}) \\ \alpha & & u & \longmapsto & \alpha \times u \end{array}$$

(which holds bc $H^*(\mathbb{R}^n, \mathbb{R}^n_0; \mathbb{Z}) \cong H^*(S^n; \mathbb{Z})$ is torsion-free)

is just

$$H^k(B;\mathbb{Z}) \cong H^k(B \times \mathbb{R}^n; \mathbb{Z}) \xrightarrow{\cup u} H^{k+n}(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0; \mathbb{Z}).$$

This establishes the result when ξ is trivial.

Now say $B = \bigcup_{i=1}^{k} U_i$ and $\xi|_{U_i} \cong U_i \times \mathbb{R}^n$ for each i.
By induction, we can assume the result holds for $\xi|_{\bigcup_{i=1}^{k-1} U_i}$.

It also holds for $\xi|_{U_k}$ and for $\xi|_{(\bigcup_1^{k-1} U_i) \cap U_k}$ b/c these last two bdles are trivial.

We now consider the relative Mayer-Vietoris sequence (letting $A' = \bigcup_1^{k-1} U_i$, $A = \bigcup_1^k U_k$)

$$H^{n-1}(\xi|_{A' \cap U_k}, (\xi|_{A' \cap U_k})_0) \to H^n(\xi|_A, (\xi|_A)_0; \mathbb{Z}) \to H^n(\xi|_{A'}, (\xi|_{A'})_0; \mathbb{Z}) \oplus H^n(\xi|_{U_k}, (\xi|_{U_k})_0; \mathbb{Z})$$
$$\longrightarrow H^n(\xi|_{A' \cap U_k}, (\xi|_{A' \cap U_k})_0; \mathbb{Z}) \longrightarrow$$

The map to the intersection term is the *difference* of the two restriction maps, so letting $u_{A'}$, u_k denote the Thom classes over $\xi|_{A'}$, $\xi|_{U_k}$, we have

$$u_{A'} \oplus u_k \longmapsto u_{A'}|_{A' \cap U_k} - u_k|_{A' \cap U_k}.$$

But since $u_{A'}$, u_k restrict to the orientation gen's on each fiber, the same is true of $u_{A'}|_{A' \cap U_k}$, $u_k|_{A' \cap U_k}$. Uniqueness of $u_{A' \cap U_k}$ then implies $u_{A'}|_{A' \cap U_k} = u_k|_{A' \cap U_k} = u_{A' \cap U_k}$. By exactness, $\exists\, u_A \in H^n(\xi|_A, (\xi|_A)_0; \mathbb{Z})$ mapping to $u_{A'} \oplus u_k$; it is unique b/c $H^{n-1}(\xi|_{A' \cap U_k}, (\xi|_{A' \cap U_k})_0) = 0$ (b/c this bdle is trivial, so thm holds). Moreover, since $A' \cup U_k = A$, u_A restricts to the orientation gen at each point of A.

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It also follows from the LES that $H^*(\xi_A, (\xi_A)_0) = 0$ for $* < n$. To examine the map $\cup u$, we use a diagram of MV seq's:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{k-1}(A'\cap U_k) \arrow[r, "\delta"] & H^k(A) \arrow[r] \arrow[d, "\cup u_A"] & H^k(A')\oplus H^k(U_k) \arrow[r] \arrow[d, "\cup u_{A'} \oplus \cup u_k"] & H^k(A'\cap U_k) \arrow[r, "\delta"] \arrow[d, "\cup u_{A'\cap U_k}"] & H^{k+1}(A) \arrow[r] & \cdots \\
& \arrow[r, "\delta"] & H^{n+k}(\xi|_A, (\xi|_A)_0) \arrow[r] & H^{n+k}(\xi|_{A'}, (\xi|_{A'})_0) \oplus H^{n+k}(\xi|_{U_k}, (\xi|_{U_k})_0) \arrow[r] & H^{n+k}(\xi|_{A'\cap U_k}, (\xi|_{A'\cap U_k})_0) \arrow[r, "\delta"] & {}
\end{tikzcd}$$

The result now follows from the 5-lemma, although one does need to check that the square involving ∂ commutes. This is similar to the issue we encountered in the proof of the Proj Bdle Thm, and uses the formulas:

$$\delta(y \cup i^*x) = \partial y \cup x \quad ; \quad i^*: H^*A'\cap U_k \to H^*A.$$

(See Lecture 11, where we proved this formula, and used it to check that a similar diagram commutes). $\square$

Note: This thm goes through w/ $\mathbb{Z}/2$ coeff's for any bdle, replacing "or. gen" with "non-zero".

There are several other ways to think about the groups $H^*(\xi, \xi_0)$: letting $D\xi = \{v \in \xi \mid |v| \le 1\}$ ("disk bdle"; refers to a chosen metric), $S\xi = \{v \in \xi \mid |v| = 1\}$ ("sphere bdle") and $T\xi = D\xi / S\xi$ (the "Thom Space" of ξ) we have

$$H^*(\xi, \xi_0) \underset{\text{excision + def. retraction}}{\cong} H^*(D\xi, S\xi) \cong H^*(T\xi)$$

$\{v \mid 1-\epsilon < |v| \le 1\}$ def. retracts to $S\xi$.

Corollary (The Gysin Sequence)

If $\begin{smallmatrix}\xi\\ \downarrow\pi\\ B\end{smallmatrix}$ is an oriented n-plane bdle over B, then there is a LES

$$\cdots \to H^i B \xrightarrow{\cup e} H^{i+n} B \to H^{i+n} \xi_0 \xrightarrow{\delta} H^{i+1} B \xrightarrow{\cup e} \cdots$$

Euler class

(w/ $\mathbb{Z}$-coeffs).

Pf: This is essentially the LES of the pair (ξ, ξ_0):

$$\to H^{i+n}(\xi,\xi_0) \to H^{i+n}(\xi) \to H^{i+n}(\xi_0) \xrightarrow{\delta} H^{i+n+1}\xi \to$$

except that we've replaced $H^{i+n}(\xi,\xi_0) \to H^{i+n}\xi$ with

Thom class

$$\begin{tikzcd} H^i B \arrow[r, "\pi^*", "\cong"'] & H^i \xi \arrow[r, "\cup u", "\cong"'] \arrow[rr, bend right, "\cup u|_\xi"'] & H^{i+n}(\xi,\xi_0) \arrow[r, "i_\xi"] & H^{i+n}(\xi) \arrow[r, "(\pi^*)^{-1}", "\cong"'] & H^{i+n} B \end{tikzcd}.$$

This composite is really just $\cup e = \cup (\pi^*)^{-1}(u|_\xi)$ b/c:

$$\begin{tikzcd} H^*\xi & H^* B \arrow[l, "\pi^*"'] \\ H^*\xi \arrow[from=1-1, "\cup u|_\xi"'] & H^* B \arrow[l, "\pi^*"'] \arrow[from=1-2, "\cup (\pi^*)^{-1}(u|_\xi)"] \end{tikzcd}$$

commutes: $\pi^*(\alpha \cup (\pi^*)^{-1} u|_\xi)$
$= \pi^*\alpha \cup u|_\xi$

So $(\pi^*)^{-1}(\pi^*\alpha \cup u|_\xi) = (\pi^*)^{-1}(\pi^*(\alpha \cup (\pi^*)^{-1} u|_\xi))$
$= \alpha \cup (\pi^*)^{-1} u|_\xi = \alpha \cup e$. $\square$

In this sequence, one often replaces ξ_0 by the sphere bdle $S\xi$, which ξ_0 deformation retracts to (again, we're assuming ξ has a metric). This LES is really the Serre Spectral Sequence for the Fibration $\begin{smallmatrix}S\xi\\ \downarrow\\ B\end{smallmatrix}$...

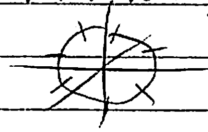
To relate the Euler class to the Euler characteristic, we need to bring in some <u>geometry</u>.

<u>Theorem</u> (MS 11.3)

Let $M^n \subset A$ be an embedded, closed submanifold of the Riemannian mfld A (that is, TA has a metric and $i: M \to A$ is a smooth homeomorphism onto its image). Then the map $H^k(A, A-M; \mathbb{Z}) \to H^k(A) \to H^k(M)$ sends a certain "Fundamental class" $u' \in H^k(A, A-M; \mathbb{Z})$ to the <u>Euler class of the normal bdle $\nu^k M = \{v \in TA \mid v \perp TM\}$</u>
$(k = \dim \nu^k)$

We'll need to describe this class u', which arises from the <u>Thom class</u> of the normal bdle of $M \subset A$. First, we need:

<u>Tubular Nbhd Thm</u>: If $M \subset A$ is a closed, embedded submfld of the Riemannian mfld A, then $\exists$ a nbhd $U \supset M$ which is <u>diffeomorphic</u> to the normal bdle $\nu^k(M) = \{v \in TA \mid v \perp TM\}$, and this diffeomorphism sends M to the zero section of $\nu^k M$.

<u>Ex</u>: $S^{n-1} \subset \mathbb{R}^n$:  U is a "spherical shell" around S^{n-1}.

We now have an excision isom

$$H^*(A, A-M;\mathbb{Z}) \cong H^*(\overset{\text{tubular nbhd}}{U}, U-M;\mathbb{Z}) \cong H^*(\nu^k M, (\nu^k M)_0;\mathbb{Z})$$

coming from excising the complement of U!! Note that the diffeomorphism $U \cong \nu^k M$ sends $M \to (\nu^k M)_0$.

We now <u>define</u> the fundamental class

$$u' \in H^*(A, A-M;\mathbb{Z})$$

to be the image of the Thom class of $\nu^k M$; here we must <u>assume</u> $\nu^k M$ is orientable.

We can now prove Thm 11.3:

<u>Pf of 11.3:</u>

We have

$$\begin{array}{ccccc} H^k(\nu^k, \nu^k_0) & \longrightarrow & H^k(\nu^k) & \xrightarrow{\overset{s = \text{zero section}}{s^*}} & H^k M \\ \underset{\text{Thom class}}{u} & \longmapsto & u|_{\nu^k} & \longmapsto & s^* u|_{\nu^k} = (\pi^*)^{-1} u|_{\nu^k} \overset{\text{def'n of Euler class}}{=} e_{\nu^k} \end{array}$$

(Note here that $\pi s = id_M$ so $s^* = (\pi^*)^{-1}$.)

So now the theorem follows immediately from the comm. diagram

$$\begin{array}{ccccc} H^k(A, A-M) & \longrightarrow & H^k(A) & \longrightarrow & H^k(M) \\ \wr\wr & & \downarrow & & \| \\ H^k(U, U-M) & \longrightarrow & H^k(U) & \xrightarrow{\cong} & H^k M \\ \wr\wr & & \wr\wr & & \| \\ H^k(\nu^k, \nu^k_0) & \longrightarrow & H^k(\nu^k) & \xrightarrow{\cong} & H^k(M) \end{array}$$

$\square$

Application to Embeddings in $\mathbb{R}^N$:

If $M^n \hookrightarrow \mathbb{R}^N$ is an (closed) embedding with orientable normal bdle ν^k ($k = N-n$) then we have shown that $e(\nu^k)$ is in the image of

$$(\star) \quad H^k(\mathbb{R}^N, \mathbb{R}^N - M) \to H^k(\mathbb{R}^N) \to H^k M.$$

But $H^k \mathbb{R}^N = 0$, so then $e(\nu^k M)$ must be zero as well.

Using Steenrod squaring op's, Milnor shows that the top Stiefel-Whitney class $w_k(\nu^k M)$ is in the image of this map $(\star)$ (w/ $\mathbb{Z}/2$ coeffs).

So then $w_k(\nu^k) = 0$. But we have $TM \oplus \nu^k M \cong T\mathbb{R}^N|_M = \varepsilon^N$,

so $w(TM) \cup w(\nu^k M) = w(\varepsilon^N) = 1$, i.e.

total Stiefel-Whitney class

$$w(\nu^k M) = w(TM)^{-1} \quad (\text{inverse in ring } H^*(M; \mathbb{Z}/2))$$

So one can solve for $w_i(\nu^k M)$ in terms of $w_i TM$, and we write $w_i(\nu^k M) = \bar{w}_i TM$ ("dual Stiefel-Whitney classes").

So if M^n embeds in $\mathbb{R}^{n+k}$, $\bar{w}_k TM = 0$ (this class depends only on M, not on the normal bdle).

Ex: $M = \mathbb{R}P^n$, $n = 2^r$. Then $\bar{w}_{n-1} T\mathbb{R}P^n = (w_1 \gamma_1^1)^{n-1} \neq 0$ (tautological bdle) $\mathbb{R}P^n$ doesn't embed in $\mathbb{R}^{2n-1}$. (We showed previously that in this case, $\mathbb{R}P^n$ doesn't immerse in $\mathbb{R}^{2n-2}$.) See Lecture 13 for the computation of $\bar{w}_{n-1}$.
Note that $\mathbb{R}P^2$ does immerse in $\mathbb{R}^3$, but (as we've just shown) doesn't embed.

So we've now related the Euler class to the normal bdle of an embedding; the next idea is:

<u>Lemma 11.5</u>: The normal bdle $\nu^n\left(M \overset{\Delta}{\hookrightarrow} M\times M\right)$ is canonically diffeomorphic to the tangent bdle of M.
$m \longmapsto (m,m)$

<u>Proof</u>: The map $D\Delta : TM \longrightarrow T(M\times M) \cong TM\times TM$ is just $v \longmapsto (v,v)$ (b/c the projections $M \xrightarrow{\Delta} M\times M \to M$ are both id_M), so we just need to show that $\nu^n \cong \{(v,v) \in T(M\times M)\}$ (note that $D\Delta : TM \to T(M\times M)$ is a bdle map, hence induces an isom. onto its image).

A vector $(u,v) \in T(M\times M)$ is normal to ΔM

$\Leftrightarrow \langle (u,v), (w,w)\rangle = 0 \quad \forall\, w \in TM$

$\Leftrightarrow 0 = \langle u, w\rangle + \langle v, w\rangle = \langle u+v, w\rangle \quad \forall\, w \in TM$

$\Leftrightarrow u+v=0.$

So $\nu^n M$ is diffeomorphic to TM via

$$(v,-v) \longmapsto (v,v). \qquad \square$$

So now 11.3 says that
$$H^k(M\times M, M\times M - \Delta M) \to H^n(M\times M) \xrightarrow{\Delta^*} H^k(M)$$
sends the "fundamental class" u' to the Euler class of TM (assuming TM is orientable). The class $u'' = u'|_{M\times M}$ will be closely related to Poincaré Duality...