Puppe Sequences:

Given a space Z and a subspace $A \subseteq Z$, we can form a sequence of inclusions

$$A \hookrightarrow Z \hookrightarrow Z \cup CA \hookrightarrow (Z \cup CA) \cup CZ \hookrightarrow \cdots$$

At each stage, we attach a cone on the subspace two steps back.

Let $\overline{Z} = Z \cup CA$. Then we can define this sequence inductively as follows:

$$A_0 = A, \quad Z_0 = Z, \quad \overline{Z}_0 = Z \cup CA.$$

We now set

$$A_i = \overline{Z}_{i-1} \cup C(Z_{i-1})$$

$$Z_i = A_i \cup C(\overline{Z}_{i-1})$$

$$\overline{Z}_i = Z_i \cup C(A_i)$$

Inductively, we have inclusions

$$\overline{Z}_{i-1} \hookrightarrow A_i \hookrightarrow Z_i \hookrightarrow \overline{Z}_i$$

so the above expressions make sense.

<u>Claim</u>: If $A \subset Z$ is a CW pair, then $A_i \xrightarrow{\simeq} S^i(A)$, $Z_i \xrightarrow{\simeq} S^i(Z)$, and $\overline{Z}_i \xrightarrow{\simeq} S^i(Z/A)$. These homotopy equivalences are natural for maps of pairs.

Proof: When $i=0$, we have $Z \cup CA \xrightarrow{q} Z/A$, where q collapses CA to a point. Since $CA \simeq *$, q is a homotopy equivalence.

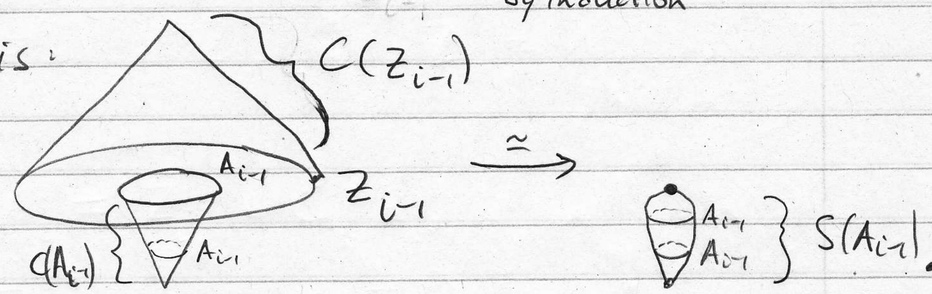
Assuming the result for $i-1$, we have

$$A_i = \bar{Z}_{i-1} \cup C(Z_{i-1}) = \left(Z_{i-1} \cup C(A_{i-1})\right) \cup C(Z_{i-1})$$

Collapsing $C(Z_{i-1})$ to a point yields a htpy equivalence

$$A_i \xrightarrow{\simeq} S(A_{i-1}) \underset{\text{by induction}}{\simeq} S(S^{i-1}A) = S^iA.$$

The picture is:



Similarly, $Z_i = A_i \cup C(\bar{Z}_{i-1}) = \left(\bar{Z}_{i-1} \cup C(Z_{i-1})\right) \cup C(\bar{Z}_{i-1})$

$$\xrightarrow[\text{collapse } C(\bar{Z}_{i-1})]{\simeq} S(Z_{i-1}) \simeq S(S^{i-1}Z) = S^iZ$$

and $\bar{Z}_i = Z_i \cup C(A_i) = \left(A_i \cup C(\bar{Z}_{i-1})\right) \cup C(A_i)$

$$\xrightarrow[\text{collapse } C(A_i)]{\simeq} S(\bar{Z}_{i-1}) \simeq S(S^{i-1}Z/A) = S^i(Z/A).$$

□

Note: the basic point here is just that for any spaces $U \subseteq W$, $[(W \cup CU) \cup CW]/CW \simeq SU$.