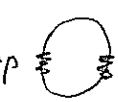
Orientability and the First Stiefel-Whitney Class:

<u>Def</u>: An <u>orientation</u> on a real vector space V is an equivalence class of bases for V, where $\{\vec{v}_1, \dots, \vec{v}_n\} \equiv \{w_1, \dots, w_n\}$ iff the transition matrix $[a_{ij}]_{ij}$ has positive determinant, where $\vec{w}_i = \sum a_{ij} \vec{v}_j$ defines the a_{ij}.

An <u>orientation</u> on a vector bundle $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ is a choice of orientation on each fiber E_b, such that there exist local trivializations
$$\begin{tikzcd} E|_U \arrow[r] \arrow[d] & U \times \mathbb{R}^n \arrow[dl] \\ U & \end{tikzcd}$$
which carry each of these orientations to the same orientation of $\mathbb{R}^n$. We'll prove:

<u>Theorem</u>: If B is locally path-connected, then $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ is orientable $\Leftrightarrow w_1(B) = 0$.

<u>Observation</u>: A bundle $\begin{smallmatrix} E \\ \downarrow \\ S^1 \end{smallmatrix}$ is trivial over the sets $S^1_+ =$  and $S^1_- =$ , so E is determined by clutching functions from the overlap  into $GL_n\mathbb{R}$. By the Bundle Homotopy Theorem, changing these clutching functions continuously does not change E.

So we may homotope these maps to constant maps. Moreover, each matrix $A \in GL_n\mathbb{R}$ can be connected by a path to either I or $\begin{bmatrix} -1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$ (i.e. $GL_n\mathbb{R}$ has two path components). Indeed, any A can be reduced to a diagonal matrix via row and column operations $A \mapsto A \cdot \begin{bmatrix} 1 & & \alpha \\ & \ddots & \\ & & 1 \end{bmatrix}$ in $(i,j)^{th}$ position and the matrices $A \cdot \begin{bmatrix} 1 & & t\alpha \\ & \ddots & \\ & & 1 \end{bmatrix}$ give a path connecting these (and similarly for row operations). This connects A to a diagonal matrix, with diagonal entries in $\mathbb{R} \setminus \{0\}$, which can all be connected to 1 or -1 by a path in $\mathbb{R} \setminus \{0\}$. Finally, if the diagonal matrix $\begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix}$ has two -1's, we can use the homotopy $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $(\theta \in [0, \pi])$ to replace these -1's by 1's.
This shows that $GL_n^+\mathbb{R} = \{A \mid \det A > 0\}$ is connected, and $GL_n^-(\mathbb{R}) \cong GL_n^+\mathbb{R}$ (multiply by $\begin{bmatrix} -1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$).
So we have at most two connected components in $GL_n\mathbb{R}$, and since $\det : GL_n\mathbb{R} \twoheadrightarrow \mathbb{R} \setminus \{0\}$ is continuous, these components must be distinct. In conclusion, the connected components of $GL_n\mathbb{R}$ are precisely $\{A \mid \det A > 0\}$, and $\{A \mid \det A < 0\}$.

Now, our bundle $\begin{matrix} E \\ \downarrow \\ S^1 \end{matrix}$ is determined by two clutching matrices, each of which is either $\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$.

If both are $\begin{bmatrix}1 & \\ & \ddots & \\ & & 1\end{bmatrix}$, E is clearly trivial, and if both are $\begin{bmatrix}-1 & \\ & \ddots & \\ & & 1\end{bmatrix}$ then E is also trivial, b/c the bases $\{e_i\}_{i=1}^n$ on S^1_+ and $\{-e_1, e_2, \dots, e_n\}$ on S^1_- give a trivialization. If one is $\begin{bmatrix}1 & \\ & \ddots & \\ & & 1\end{bmatrix}$ and the other is $\begin{bmatrix}-1 & \\ & \ddots & \\ & & 1\end{bmatrix}$, then $E \cong \gamma_1^1 \oplus \varepsilon^1 \oplus \dots \oplus \varepsilon^1$. To understand orientability of E, need:

<u>Lemma</u>: $\begin{matrix}\mathbb{R}^n \to E \\ \downarrow \\ B\end{matrix}$ is orientable $\iff$ $\begin{matrix}\det(E) \\ \downarrow \\ B\end{matrix}$ is orientable,

where $\det(E) = \Lambda^n E$ is the determinant line bundle.

<u>Pf</u>: If E is orientable, then choose an orientation $\{v_i(b)\}_{i=1}^n$. The orientation class of $v_1(b) \wedge \dots \wedge v_n(b) \in \det(E)_b$ depends only on the orientation class of the basis, because for any $A \in GL_n\mathbb{R}$, and any $\vec{w}_i \in \mathbb{R}^n$, $Aw_1 \wedge \dots \wedge Aw_n = \det A \, w_1 \wedge \dots \wedge w_n$: When $w_i = e_i = \begin{pmatrix}0\\ \vdots \\ 1 \\ \vdots \\ 0\end{pmatrix}$, then $Ae_1 \wedge \dots \wedge Ae_n = \begin{pmatrix}a_{11}\\ \vdots \\ a_{n1}\end{pmatrix} \wedge \dots \wedge \begin{pmatrix}a_{1n}\\ \vdots \\ a_{nn}\end{pmatrix} = \det(A)\, e_1 \wedge \dots \wedge e_n$ by cofactor expansion. Then in general we set $W = (\vec{w}_1 \cdots \vec{w}_n)$ and have:

$$Aw_1 \wedge \dots \wedge Aw_n = AWe_1 \wedge \dots \wedge AWe_n = \det(AW) e_1 \wedge \dots \wedge e_n$$
$$= \det A \det W \, e_1 \wedge \dots \wedge e_n = \det(A)\, w_1 \wedge \dots \wedge w_n$$

as claimed. So changing the $v_i(b)$ by a positive determinant matrix just changes $v_1(b) \wedge \dots \wedge v_n(b)$ by a positive number. If the $\{v_i(b)\}$ are locally consistent, so are the bases $v_1(b) \wedge \dots \wedge v_n(b)$ of $\det(E)$.

Conversely, if $\begin{matrix}\det(E)\\ \downarrow\\ B\end{matrix}$ is orientable, then an orientation $v_1(b) \wedge \cdots \wedge v_n(b)$ determines an orientation $\{v_1(b), \cdots, v_n(b)\}$ in exactly the same manner. ! $\square$

<u>Lemma</u>: A line bdle $\begin{matrix}L\\ \downarrow\\ B\end{matrix}$ over a paracompact space is orientable $\Longleftrightarrow$ it is trivial.

<u>Pf</u>: $\Leftarrow$: immediate

$\Rightarrow$: If L is orientable and we choose a metric on L, then the unique unit vector in the given orientation class provides a continuous, nowhere zero section of L. $\square$

<u>Claim</u>: $E = \gamma_1^1 \oplus \varepsilon^1 \oplus \varepsilon^1$ is <u>not</u> orientable.

<u>Pf</u>: γ_1^1 is not orientable b/c it's non-trivial. Now, $\det(\gamma_1^1 \oplus \varepsilon^1 \oplus \cdots \oplus \varepsilon^1) \cong \gamma_1^1 \otimes \varepsilon^1 \otimes \cdots \otimes \varepsilon^1 \cong \gamma_1^1$, so $\det(\gamma_1^1 \oplus \varepsilon^1 \oplus \cdots \oplus \varepsilon^1)$ is not orientable, and neither is $\gamma_1^1 \oplus \varepsilon^1 \oplus \cdots \oplus \varepsilon^1 = \gamma_1^1 \oplus \varepsilon^{n-1}$. $\square$

<u>Corollary</u>: Over the circle, there are precisely two bdles of each dimension, and the non-trivial one is non-orientable.

To better understand oriented bundles, we can consider their associated principal bundles.

If $\begin{matrix}E\\ \downarrow\\ B\end{matrix}$ has an orientation, then $\begin{matrix}Fr^+E\\ \downarrow\\ B\end{matrix}$, the bundle of <u>oriented</u> frames (meaning frames $\{v_1, \dots, v_n\} \subseteq E_b$ lying in the correct orientation class) is a principal $GL_n^+\mathbb{R} = \{A \mid \det A > 0\}$-bundle (just as $Fr(E)$ is a $GL_n\mathbb{R}$-bdle).

Conversely, given a principal $GL_n^+\mathbb{R}$-bdle $\begin{array}{c} P \\ \downarrow \\ B \end{array}$, the mixed bundle $\begin{array}{c} P \times_{GL_n^+\mathbb{R}} \mathbb{R}^n \\ \downarrow \\ B \end{array}$ will be an <u>oriented</u> vector bundle. This provides a correspondence b/w oriented bdles over B and $GL_n^+\mathbb{R}$-bdles over B.

The latter are classified by maps $B \to Gr_n^+\mathbb{R}^\infty$, where $Gr_n^+\mathbb{R}^\infty = V_n(\mathbb{R}^\infty) \Big/ \left(\begin{array}{c} \{v_1, \dots, v_n\} \sim \{v_1, \dots, v_n\} \cdot A \\ \text{for all } A \in GL_n^+\mathbb{R} \end{array} \right)$

This is called the <u>oriented</u> Grassmanian, and it's a universal $GL_n^+\mathbb{R}$-bundle, b/c $\pi_* V_n\mathbb{R}^\infty = 0$ and slices can be constructed in a similar manner as we did in the $GL_n\mathbb{R}$-case.

<u>Claim</u>: $Gr_n^+\mathbb{R}^\infty$ is the universal cover of $GL_n(\mathbb{R}^\infty)$, and $\pi_1(GL_n\mathbb{R}^\infty) = \mathbb{Z}/2$.

<u>Pf</u>: To show $\pi_1 Gr_n^+\mathbb{R}^\infty = 0$, just need to show $[S^1, Gr_n^+\mathbb{R}^\infty] = 0$. But these htpy classes correspond bijectively with oriented

bundles over S^1, and we've shown that there's only one of these (the trivial bundle).

Now, the map $Gr_n^+ \mathbb{R}^\infty \to Gr_n \mathbb{R}^\infty$ just identifies an orientation class of frames with its opposite orientation class, so it's a free action of $\mathbb{Z}/2$. The Grassmanian is Hausdorff (see MS Chapter 5, Lemma 5.1), so this is automatically a covering space action. Hence $\pi_1(Gr_n \mathbb{R}^\infty) = \mathbb{Z}/2$. $\square$

<u>Theorem</u>: $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is orientable $\iff w_1(E) = 0$, if B is paracompact and locally connected.

<u>Pf</u>: $\Rightarrow$ If E is orientable, then it's orientable after pulling back along any loop. Orientable bdles over S^1 are trivial, so $\gamma^* E \simeq S^1 \times \mathbb{R}$ for all $S^1 \to B$. By our def'n of $w_1(E)$, we see that $w_1(E) = 0$.

$\Leftarrow$ If $w_1 E = 0$, then $\gamma^* E$ is trivial for all $\gamma : S^1 \to E$. So the classifying map for $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$, $B \xrightarrow{f} Gr_n \mathbb{R}^\infty$, induces a trivial map on π_1. From basic covering space theory, this means f lifts to $Gr_n^+(\mathbb{R}^\infty) =$ univ. cover of $Gr_n \mathbb{R}^\infty$:

$$\begin{tikzcd} & Gr_n^+ \mathbb{R}^\infty \arrow[d, "\pi"] \\ B \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & Gr_n \mathbb{R}^\infty \end{tikzcd}$$

So $E = f^* V_n\mathbb{R}^\infty = \tilde{f}^* \pi^*\left(V_n\mathbb{R}^\infty \times_{GL_n\mathbb{R}} \mathbb{R}^n\right) = \tilde{f}^*\left(V_n\mathbb{R}^\infty \times_{GL_n^+\mathbb{R}} \mathbb{R}^n\right)$

b/c the diagram

$$\begin{array}{ccc} V_n\mathbb{R}^\infty \times_{GL_n^+\mathbb{R}} \mathbb{R}^n & \longrightarrow & V_n\mathbb{R}^\infty \times_{GL_n\mathbb{R}} \mathbb{R}^n \\ \downarrow & & \downarrow \\ Gr_n^+\mathbb{R}^\infty & \longrightarrow & Gr_n\mathbb{R}^\infty \end{array}$$

is a pullback diagram.

So E is pulled back from $V_n\mathbb{R}^\infty \times_{GL_n^+\mathbb{R}} \mathbb{R}^n$, which has a canonical orientation, so E is oriented. $\square$