Characteristic Classes as Obstructions:

Given a CW cplx B and a bdle $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$,

the first Stiefel-Whitney class $w_1(E) \in H^1(B; \mathbb{Z}/2)$ can be constructed as follows:

For each vertex $x \in B$, choose a basis for E_x. Now, given a 1-cell, $\overset{\sigma^1}{x \frown y}$, consider the pullback of E along the characteristic map for σ^1: $\begin{matrix} \varphi^* E \\ \downarrow \\ [0,1] \end{matrix}$. We know that $\varphi^* E$ is trivial, b/c $[0,1] \simeq *$. If $f: \varphi^* E \xrightarrow{\cong} [0,1] \times \mathbb{R}^n$ is a trivialization, then on either end of $[0,1]$ we can compare $f^{-1}\{e_1, \dots, e_n\}$ to our chosen bases for E_x and E_y:

$$\begin{tikzcd}
\tilde{\varphi}^{-1}\{v_1(x), \dots, v_n(x)\} \sim & \{v_1(x), \dots, v_n(x)\} \arrow[d, phantom, "\cap" rotate=90] & \\
(\varphi^* E)_x \arrow[r, "\tilde{\varphi}"] \arrow[d] & E_x \arrow[d] & E_y \arrow[d] \\
{[0,1]} \arrow[r, "\varphi"] & x \overset{\sigma^1}{\frown} y &
\end{tikzcd}$$

The differences b/w these bases are measured by matrices $A_x, A_y \in GL_n \mathbb{R}$. Multiplying φ by A_x^{-1} at each pt, we may assume $A_x = I$.

So we get a based map $\partial[0,1] = S^0 \longrightarrow GL_n \mathbb{R}$, which is simply an elt in $\pi_0(GL_n\mathbb{R}) \cong \mathbb{Z}/2$.

Thus we have assigned, to each 1-cell $\sigma^1 \subseteq B$, an elt of $\pi_0 GL_n\mathbb{R} \cong \mathbb{Z}/2$. This is a $\mathbb{Z}/2$-valued cocycle, and in fact represents the cohomology class

$$w_1(E) \in H^1(B;\mathbb{Z}/2).$$

Now, let's say $w_1(E)=0$, i.e. E is trivial along each loop in B. Then in particular, E becomes trivial when restricted to the 1-skeleton of B:

The one-skeleton is always homotopy equivalent to a wedge of circles, and we can adjust the trivializations over the circles so that they agree at the wedge point.

Let's consider whether E is trivial over the 2-skeleton of B. Given a 2-cell w/ characteristic map $\varphi: D^2 \to B$, we know that

$\begin{matrix}\varphi^*E\\ \downarrow\\ D^2\end{matrix}$ is trivial. Say we've chosen a trivialization of $E|_{B(1)}$. Then this gives an orientation of φ^*E along $\partial D^2 = S^1$. If $f: \varphi^*E \to D^2 \times \mathbb{R}^n$ is a trivialization, f may carry the orientation on $\varphi^*E|_{\partial D^2}$ to either the standard orient'n of $\mathbb{R}^n$ or the opposite orientation; by composing with the map (mult'n by $\begin{bmatrix}-1 & & \\ & 1 & \\ & & \ddots\end{bmatrix}$) if necessary, we make f respect the orientations.

Now, at each point in $\partial D^2 = S^1$, we can consider the difference between our trivialization on $E|_{\partial D^2}$, coming from our trivialization of $E|_{B(1)}$, and our trivialization of $\begin{matrix}\varphi^*E\\ \downarrow\\ D^2\end{matrix}$. Since the bases determined by these trivializations are in the same orientation class, at each point the transition matrix lies in $GL_n^+\mathbb{R}$ and we obtain a map $S^1 \to GL_n^+\mathbb{R} \overset{\text{Gram-Schmidt}}{\simeq} SO(n)$.

This data determines a $\pi_1 SO(n)$-valued 2-cycle, representing a class
$$\tilde{w}_2(E) \in H^2(B; \pi_1 SO(n)),$$
which will vanish $\Longleftrightarrow$ $E|_{B^{(2)}}$ is trivial.

For $n=1$, $SO(1) = \{1\}$ and $w_2 \equiv 0$ (this is the case of a line bdle). When $n=2$, $SO(2) \cong S^1$ and $\pi_1 SO(2) \cong \mathbb{Z}$. The class $w_2(E) \in H^2(B; \mathbb{Z}/2)$ is the mod-2 reduction of $\tilde{w}_2(E)$. For $n \geq 3$, $\pi_1 SO(n) = \mathbb{Z}/2$ and $\tilde{w}_2(E) = w_2(E)$.

To construct the higher w_k, it's better to consider the question of existence of a <u>single</u> section of E, rather than a trivialization (n linearly indep. sections).

Say E is **orientable, and** we have a section of E over the k-skeleton. To extend this to a section over $B^{(k+1)}$, we need to extend it across each $(k+1)$-cell.

This means we have a bdle $\begin{matrix} E \cong D^{k+1} \times \mathbb{R}^n \\ \downarrow \\ D^{k+1} \end{matrix}$ and a section on $S^k = \partial D^{k+1}$. Choosing a metric, we can assume that our section lies in $S^{n-1} \subseteq \mathbb{R}^n$ at each point, and we just need to know if this map

$$S^k \longrightarrow S^{n-1}$$

is null homotopic. [Our orientation lets us consistently identify the spheres in different fibers, at least up to an elt of $SO(n)$.]

So we can assign, to each $(k+1)$-cell, this elt in $\pi_k(S^{n-1})$.

Since $\pi_k S^{n-1} = 0$ for $k < n-1$, we can extend our section over the $n-2$ skeleton, and when we hit the $(n-1)$-skeleton, we obtain a cohomology class in

$$H^{n-1}(B; \pi_{n-1} S^{n-1}) = H^{n-1}(B; \mathbb{Z}).$$

The mod-2 reduction of this class is precisely $w_n(E)$.

So for an orientable n-plane bdle, w_n is the obstruction to existence of a section on the $(n-1)$-skeleton.

Similarly, $w_{n-k+1} E$ is the obstruction to finding k o.n. sections over the $(n-k+1)$-skeleton.