

Bott Periodicity II: Reduction to Laurent Poly. Generating Funs

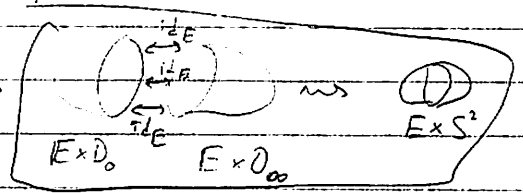
We are trying to show that the map

$$\begin{array}{ccccc} K(X) \otimes \mathbb{Z}[H] / (H-1)^2 & \longrightarrow & K(X) \otimes K(S^2) & \longrightarrow & K(X \times S^2) \\ E \otimes 1 & \longmapsto & E \otimes 1 & \longmapsto & \pi_1^* E \\ E \otimes H & \longmapsto & E \otimes H & \longmapsto & \pi_1^* E \otimes \pi_2^* H \end{array}$$

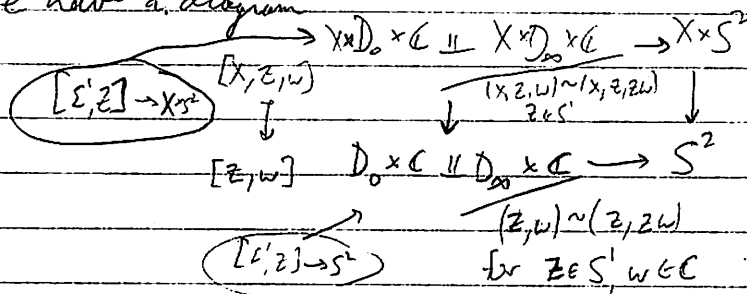
is an isomorphism (when X is cpt Hausdorff). On

the RHS, $\pi_1^* E = [E, \mathbb{1}]$

and $\pi_2^*(H) \cong [\mathcal{E}', \mathbb{Z}]$:



to see the second isom, recall that $H = [\mathcal{E}', \mathbb{Z}]$ as a bundle over S^2 , and we have a diagram



which is easily seen to be a pullback diagram.

Similarly, $\pi_2^* H^m \cong \pi_2^* [\mathcal{E}', \mathbb{Z}^m] = [\mathcal{E}', \mathbb{Z}^m]$ (where on the RHS $\mathcal{E}' = \begin{array}{c} X \times C \\ \downarrow \\ X \end{array}$),

for any $m \in \mathbb{Z}$.

So to prove surjectivity, it will suffice to show that every bundle $E' = [E, F]$ over $X \times S^2$ has the form

$\bigoplus_i [E_i, \mathbb{Z}^{m_i}]$ for some $m_i \in \mathbb{Z}$. (Note: it will then follow

that the bundles $[E, \mathbb{Z}], [E, \mathbb{1}]$ generate $K(X \times S^2)$ additively.)

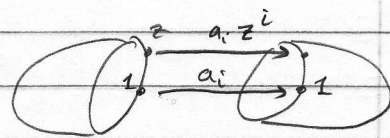
First Step: replace $f: E \times S^1 \xrightarrow{\cong} E \times S^1$ by a

Laurent poly clutching Function $l = \sum_{i=-N}^N a_i z^i$,

i.e. $l(e, z) = \sum_{i=-N}^N a_i(e) z^i$

for some endomorphisms: $E \xrightarrow{a_i} E$

$a_i(x): E_x \rightarrow E_x$
 is linear for each x ,
 but not necessarily an isomorphism.



clutching for $l = \sum a_i z^i$.

Propn 2.4: For each $f: E \times S^1 \xrightarrow{\cong} E \times S^1$, $\exists l = \sum_{i=-N}^N a_i z^i$

s.t. $[E, f] \cong [E, l]$. Moreover, if l_0 and l_1 are

Laurent poly clutching fcn's that are homotopic through clutching fcn's, then they're homotopic through a Laurent poly clutching fcn homotopy

Laurent poly clutching fcn homotopy $l(x, z) = l(x, t, z) = \sum_{i=-N}^N a_i(x, t) z^i$

(w/ a_i an endomorphism $E \times I \xrightarrow{a_i} E \times I$)

Locally, $f: E \times S^1 \rightarrow E \times S^1$ can be viewed as a family of linear isoms $S^1 \times U \times \mathbb{C}^n \rightarrow S^1 \times U \times \mathbb{C}^n$; i.e. $S^1 \times U \rightarrow GL_n \mathbb{C}$.

We will show that fcn's $S^1 \times X \rightarrow GL_n \mathbb{C}$ can be approximated by Laurent poly's - more specifically, by the partial sums of their Fourier series - and that if $l \approx f$ then

- 1) l is still a clutching fcn (i.e. total automorphism)
- 2) there is a "linear" homotopy $t l + (1-t) f$ from l to f which stays inside the space of clutching fcn's.

These steps will follow from open-ness and local convexity of the space of clutching fcn's, considered as a subset of the vector space $\text{End}(E \times S^1)$

in the norm $\|\alpha\| = \sup_{|M|=1} |\alpha(V)|$ (where $| \cdot |$ denotes some metric on $E \times S^1$).

We now show that locally F can be approximated by Laurent poly's, starting with the case of a single fn $X \times S^1 \rightarrow \mathbb{C}$

Lemma 2.5: For any $f: X \times S^1 \rightarrow \mathbb{C}$, let

$$a_n(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x, e^{it}) e^{-int} dt$$

denote the Fourier coeff's of F . Then as $r \rightarrow 1$, the infinite sum

$$u(x, r, \theta) = \sum_{n \in \mathbb{Z}} a_n(x) r^{|n|} e^{in\theta}$$

converges uniformly (in x and θ) to F . In other words,

$\forall \epsilon > 0 \exists \delta > 0$ st. if $r \in (1-\delta, 1)$ then for all $(x, \theta) \in X \times [0, 2\pi]$

$$|F(x, e^{i\theta}) - u(x, r, \theta)| < \epsilon.$$

Since the sum is convergent, it follows that $\forall \epsilon > 0 \exists \delta > 0, N \in \mathbb{N}$ st. $|F(x, e^{i\theta}) - \sum_{n=-N}^N a_n(x) r^{|n|} e^{in\theta}| < \epsilon$ for all $r \in (1-\delta, 1)$ and all $(x, \theta) \in X \times [0, 2\pi]$.

Proof: First we check that $\sum_{n \in \mathbb{Z}} a_n(x) r^{|n|} e^{in\theta}$ is convergent, so that u is well-defined. We have:

$$|a_n(x)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(x, e^{it}) e^{-int} dt \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(x, e^{it})| dt \leq C$$

where C is a bound on the cts for $f(x, z): X \times S^1 \rightarrow \mathbb{C}$ (note that X is compact, so a bd exists). Hence $|a_n(x) r^{|n|} e^{in\theta}| \leq C r^{|n|}$,

and $\sum a_n(x) r^{|n|} e^{in\theta}$ converges (for $r < 1$) by comparison w/ $\sum r^{|n|}$.

Now,

$$u(x, r, \theta) = \sum_{n=-\infty}^{\infty} a_n(x) r^{|n|} e^{in\theta} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} f(x, e^{it}) e^{-int} dt \right) r^{|n|} e^{in\theta}$$

$$= \sum_{n=-\infty}^{\infty} \int_0^{2\pi} \left(\frac{1}{2\pi} f(x, e^{it}) |r|^n e^{in(\theta-t)} dt \right)$$

$$= \int_0^{2\pi} \left(\sum_{n=-\infty}^{\infty} \frac{1}{2\pi} f(x, e^{it}) |r|^n e^{in(\theta-t)} \right) dt$$

b/c sum

converges uniformly

in t , by some argument as above.

$$= \int_0^{2\pi} f(x, e^{it}) \underbrace{\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)}}_{P(r, \varphi) \text{ (Poisson Kernel)}}$$

$$= \int_0^{2\pi} f(x, e^{it}) \cdot P(r, \theta-t) dt.$$

Claim 1: $P(r, \varphi) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos\varphi+r^2}$, so $P(r, \varphi) > 0$ for all $0 < r < 1, \varphi \in \mathbb{R}$.

$$\text{PF: } 2\pi P(r, \varphi) = \sum_{n=1}^{\infty} r^n e^{-in\varphi} + \sum_{n=0}^{\infty} r^n e^{in\varphi}$$

$$= \frac{re^{-i\varphi}}{1-re^{-i\varphi}} + \frac{1}{1-re^{i\varphi}} = \frac{(1-re^{i\varphi})re^{-i\varphi} + (1-re^{-i\varphi})}{(1-re^{-i\varphi})(1-re^{i\varphi})}$$

$$= \frac{\cancel{re^{-i\varphi}} - r^2 + 1 - \cancel{re^{i\varphi}}}{1 - \underbrace{re^{-i\varphi} - re^{i\varphi}}_{e^{-i\varphi} + e^{i\varphi}} + r^2} = \frac{1-r^2}{1-r^2\cos\varphi+r^2}$$

$$e^{-i\varphi} + e^{i\varphi} = \cos\varphi + i\sin\varphi + \cos\varphi + i\sin\varphi = 2\cos\varphi$$

So $P(r, -)$ is 2π -periodic, even, monotone decreasing on $[0, \pi]$, and monotone increasing on $[\pi, 2\pi]$.

Claim 2: $\int_a^{a+2\pi} P(r, \varphi) d\varphi = 1$ for any $r < 1$.

$$\text{PF: } \int_a^{a+2\pi} P(r, \varphi) d\varphi = \int_a^{a+2\pi} \left(\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\varphi} \right) d\varphi = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} \int_a^{a+2\pi} e^{in\varphi} d\varphi$$

$$= \frac{1}{2\pi} \sum_{n \neq 0} r^{|n|} \frac{e^{in\varphi}}{in} \Big|_a^{a+2\pi} + \frac{1}{2\pi} \int_a^{a+2\pi} 1 d\varphi = 1 \quad \square$$

Now we examine the difference between F and u :

$$|u(x, r, \theta) - f(x, e^{i\theta})| = \left| \int_0^{2\pi} P(r, \theta-t) f(x, e^{it}) dt - f(x, e^{i\theta}) \cdot \int_0^{2\pi} P(r, \theta-t) dt \right|$$

1 by claim 2

$$= \left| \int_0^{2\pi} P(r, \theta-t) (f(x, e^{it}) - f(x, e^{i\theta})) dt \right|$$

$$\leq \int_0^{2\pi} P(r, \theta-t) |f(x, e^{it}) - f(x, e^{i\theta})| dt$$

Note $P > 0$ we'll break this integral into 2 pieces.

Since $F: X \times S^1 \rightarrow \mathbb{C}$ is continuous and X is cpt,

given $\epsilon > 0$, $\exists \delta_1 > 0$ ($\delta_1 < \frac{\pi}{2}$) s.t. $|\theta - t| < \delta_1 \Rightarrow |f(x, e^{i\theta}) - f(x, e^{it})| < \epsilon$
for all $x \in X$

Let $I_{\delta_1} = \int_{\theta - \delta_1}^{\theta + \delta_1} P(r, \theta-t) |f(x, e^{it}) - f(x, e^{i\theta})| dt$, so that

$$0 \leq I_{\delta_1} \leq \int_{\theta - \delta_1}^{\theta + \delta_1} P(r, \theta-t) \epsilon dt \leq \left(\int_{\theta - \delta_1}^{\theta - \delta_1 + 2\pi} P(r, \theta-t) dt \right) \epsilon = \epsilon$$

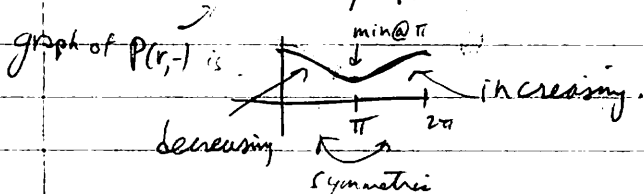
On the other hand,

$$I'_{\delta_1} \stackrel{\text{def}}{=} \int_{\theta + \delta_1}^{\theta - \delta_1 + 2\pi} P(r, \theta-t) |f(x, e^{it}) - f(x, e^{i\theta})| dt = \int_{\theta + \delta_1}^{\theta - \delta_1 + 2\pi} P(r, \varphi) \cdot 2C d\varphi$$

$|f(x, z)| < C$
for $(x, z) \in X \times S^1$

$$\varphi = \theta - \delta_1, \quad P(r, \varphi) = P(r, -\varphi)$$

$$\leq 2C \cdot P(r, \delta_1)$$



Now, as $r \rightarrow 1$; $P(r, \delta_1) = \frac{1-r^2}{1-2r \cos \delta_1 + r^2} \rightarrow \frac{0}{2-2 \cos \delta_1} = 0$,
 $\cos \delta_1 \neq 1$

for $\exists \delta$ s.t. $r \in (1-\delta, 1) \Rightarrow P(r, \delta_1) < \frac{\epsilon}{2C}$.

Then $|u(x, r, \theta) - f(x, \theta)| \leq I_{\delta_1} + I'_{\delta_1} \leq \epsilon + 2C \frac{\epsilon}{2C} = 2\epsilon$ for $r \in (1-\delta, 1)$. \square

Proof of Prop'n 2.4: Choose a Hermitian metric $\langle \cdot, \cdot \rangle$

on E . Then $\text{End}(E \times S^1) =$ set of ^{self} endomorphisms is

a vector space under ptwise addition and scalar multn, and

in fact it admits a norm: $\|\alpha\| = \sup_{\|v\|=1} |\alpha(v)|$. $\leftarrow \|\alpha(v)\| = \sqrt{\langle \alpha(v), \alpha(v) \rangle}$

We claim that

1) Balls in the $\|\cdot\|$ norm are convex: say $t \in (0,1)$.

If $\|\alpha - \gamma\|, \|\beta - \gamma\| < \varepsilon$, then $\|(t\alpha + (1-t)\beta) - \gamma\| < \varepsilon$;

2) $\text{Aut}(E \times S^1) \subseteq \text{End}(E \times S^1)$ is open.

Assuming 1) and 2), we now check that every

bdle $[E, F]$ is isomorphic to $[E, \mathbb{C}]$ for some Laurent

poly. clutching fcn $h(x, z) = \sum_{n=-N}^N a_n(x) z^n$ with $a_n(x) \in \text{End}(E)$.

Since $F \in \text{Aut}(E \times S^1)$ and $\text{Aut}(E \times S^1) \subseteq \text{End}(E \times S^1)$

is open, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(F) \subseteq \text{Aut}(E \times S^1)$. Moreover,

if $g \in B_\varepsilon(F)$ is any other ~~the~~ bdle automorphism, then for $t \in (0,1)$,

$tF + (1-t)g \in B_\varepsilon(F)$ by 2), so $tF + (1-t)g$ is

a type of clutching fcn connecting F to g , so $[E, F] \cong [E, g]$.

So we will just need to show that Laurent poly's are dense in $\text{End}(E \times S^1)$.

The proof of 1) is simple: $\|\alpha + \beta\| = \sup_{\|v\|=1} |\alpha(v) + \beta(v)| \leq \sup_{\|v\|=1} |\alpha(v)| + \sup_{\|v\|=1} |\beta(v)|$

$\leq \sup_{\|v\|=1} |\alpha(v)| + \sup_{\|v\|=1} |\beta(v)| = \|\alpha\| + \|\beta\|$, i.e. $\|\cdot\|$ satisfies the Δ -ineq.

Now $\|(t\alpha + (1-t)\beta) - \gamma\| = \|(t\alpha - t\gamma) + ((1-t)\beta - (1-t)\gamma)\| \leq \|t(\alpha - \gamma)\| + \|(1-t)(\beta - \gamma)\| \leq t\|\alpha - \gamma\| + (1-t)\|\beta - \gamma\| < \varepsilon t + (1-t)\varepsilon = \varepsilon$.

To prove 2), it suffices to show that

$$\inf_{(x,z)} |\det(\alpha_{(x,z)}: E_x \times \{z\} \rightarrow E_x \times \{z\})|: \text{End}(E \times S^1) \rightarrow [0, \infty)$$

is continuous; then $\text{Aut}(E \times S^1)$ is the inverse image of $(0, \infty)$.

in the metric topology induced by $\|\cdot\|$

First note that $\det(\alpha_{(x,z)})$ is well-defined: it's independent

of any chosen basis for $E_x \times \{z\}$ (bc $\det(PAP^{-1}) = \det(A)$).

Now, say $\|\alpha - \beta\| < \delta$. Then if we choose an orthonormal basis $\{e_j\}$ for $E_x \times \{z\}$, α and β are rep'd by matrices $A = (a_{ij})$ and $B = (b_{ij})$.

Since $(A-B)e_i = \sum (a_{ij} - b_{ij})e_j$ and $\|e_i\| = 1$, we have

$$\left| \sum (a_{ij} - b_{ij})e_j \right| \leq \sup_{\|v\|=1, v \in E_x \times \{z\}} |(A-B)v| \leq \sup_{\|v\|=1, v \in E \times S^1} |(\alpha - \beta)v| = \|\alpha - \beta\| < \delta$$

But $\langle e_i, e_j \rangle = \delta_{ij}$ or $\left| \sum (a_{ij} - b_{ij})e_j \right| = \sqrt{\sum |a_{ij} - b_{ij}|^2}$.
Kronecker function

Thus $|a_{ij} - b_{ij}| < \delta \forall i, j$, meaning the matrices

A and B are element-wise close. Since $\text{Det}: M_n(\mathbb{C}) \rightarrow [0, \infty)$

is continuous, $\forall \epsilon > 0 \exists \delta > 0$ s.t. $|a_{ij} - b_{ij}| < \delta \forall i, j \Rightarrow |\det A$

$$\Rightarrow |\det A - \det B| < \epsilon.$$

Now if $\|\alpha - \beta\| < \delta$, then we have $\left| |\det \alpha_{(x,z)}| - |\det \beta_{(x,z)}| \right| < \epsilon$

for every $(x,z) \in E \times S^1$. Hence for some $(x_0, z_0), (x_1, z_1)$

$$\inf |\det \alpha| \geq |\det \alpha_{(x_0, z_0)}| - \epsilon \geq |\det \beta_{(x_0, z_0)}| + 2\epsilon \geq \inf |\det \beta| - 2\epsilon$$

$$\text{and } \inf |\det \beta| \geq |\det \beta_{(x_1, z_1)}| - \epsilon \geq |\det \alpha_{(x_1, z_1)}| - 2\epsilon \geq \inf |\det \alpha| - 2\epsilon$$

So $\|\alpha - \beta\| < \delta \Rightarrow \inf |\det \alpha| - \inf |\det \beta| \leq 2\epsilon.$

Finally, we must apply the Lemma to show that Laurent poly's are dense in $(\text{End}(E \times S^1), \|\cdot\|)$.

Choose a finite open covering $\{U_i\}$ of X such that there exist isometric trivializations $h_i: E|_{U_i \times S^1} \xrightarrow{\cong} U_i \times S^1 \times \mathbb{C}^{n_i}$. E may have different dim's over different comp's of X Choose a part of $\mathcal{I} \{ \varphi_i \}$ subordinate to $\{U_i\}$, with $X_i = \text{supp}(\varphi_i)$ (a cpt set inside U_i). Let $\alpha_i = \alpha|_{X_i \times S^1}$, and set $A_i = h_i \alpha_i h_i^{-1}: X_i \times S^1 \times \mathbb{C}^{n_i} \rightarrow \mathbb{C}^{n_i}$.

Then A_i is a map $X_i \times S^1 \rightarrow M_{n_i} \mathbb{C}$, and we can choose

Laurent poly's $(l_i)_{k,l}: X_i \times S^1 \rightarrow \mathbb{C}$ which approximate the entries of A_i to within ε (uniformly over $X_i \times S^1$).

Let $l_i = (l_i)_{k,l}: X_i \times S^1 \times \mathbb{C}^{n_i} \rightarrow \mathbb{C}$, and by abuse of notation set $l_i = h_i^{-1} l_i h_i: E|_{X_i \times S^1} \rightarrow \mathbb{C}$.

$$\text{Then } \|\alpha - \sum_j \varphi_j l_j\| = \sup_{|v|=1} |\alpha v - \sum_j \varphi_j l_j v|$$

Still 2 Laurent poly: $l_j = \sum_{-N_j}^{N_j} a_{j,n} |z|^{n_j}$
 $\Rightarrow \sum_j \varphi_j l_j = \sum (\varphi_j a_{j,n}) |z|^{n_j}$

$$\begin{aligned} &= \sup_{|v|=1} |\sum_j \varphi_j \alpha v - \sum_j \varphi_j l_j v| \leq \sup_{|v|=1} \sum_{j: v \in E|_{X_j}} |\alpha v - l_j v| \\ &\leq N \cdot \varepsilon \end{aligned}$$

But if $v = \sum v_k e_k$ (e_k an o.n. basis for fiber containing v)

then $|v_k| \leq 1$ for each k and $|(\alpha + l_j)(\sum v_k e_k)| \leq \sum |\alpha - l_j| |v_k|$

$$= \sum (|A_{i,ik} - (l_j)_{ik}|) |v_k| \leq n_i^2 \cdot \varepsilon.$$

\uparrow
 (i,k) entries

So $\|\alpha - \sum \varphi_j l_j\| \leq n_i^2 \cdot \varepsilon$. So for ε small, $\sum \varphi_j l_j$ is invertible and homotopic to α . For $X = \text{char poly}$, we won't need the part about Laurent homotopies. \square