Constructions on Bundles (MS Chapter 3)

To justify these arguments, we need to explain the general procedure for constructing bundles like

$$Hom(E_1, E_2), \quad Hom(E, \mathbb{R}) = E^*,$$
$$E_1 \otimes E_2, \text{ etc.}$$

We'll follow MS Chapter 3, which gives a general procedure for extending "continuous" operations on vector spaces to operations on vector bundles.

Def'n: Let Vect denote the category of finite dimensional vector spaces (over $\mathbb{R}$ or $\mathbb{C}$) and linear isomorphisms. A functor $F: (\underline{Vect})^k \longrightarrow \underline{Vect}$ (taking a list of k vector spaces to a single vector space) is continuous if the function on morphisms

$$\begin{gathered} Isom(V_1, W_1) \times Isom(V_2, W_2) \times \cdots \times Isom(V_k, W_k) \\ \downarrow F \\ Isom(F(V_1, \ldots, V_k), F(W_1, \ldots, W_k)) \end{gathered}$$

is continuous.

Here $Isom(V, W) \subseteq Hom(V, W)$ is given the subspace topology. The topology on $Hom(V, W)$ can be thought about in either of two equivalent ways.

First, <u>every</u> vector space V has a canonical topology, obtained by choosing an isomorphism

$$\varphi : V \xrightarrow{\cong} \mathbb{R}^n$$

and declaring $U \subseteq V$ to be open $\Leftrightarrow$ $\varphi(U) \subseteq \mathbb{R}^n$ is open. This topology is independent of the choice of isomorphism φ, b/c if we have

$$\mathbb{R}^n \xleftarrow[\psi]{\cong} V \xrightarrow[\varphi]{\cong} \mathbb{R}^n$$

then $\psi \circ \varphi^{-1}$ is a (linear) homeomorphism, so $\varphi(U)$ is open $\Leftrightarrow$ $(\psi \circ \varphi^{-1})(\varphi(U)) = \psi(U)$ is open.

Now we can topologize $Hom(V, W)$ either by observing that it's a vector space in its own right, or by giving it the <u>compact-open</u> topology inherited from the mapping space $Map(V, W)$. The first topology is useful when checking that functors like $\otimes$ and Hom are continuous, but the second has good general properties.

<u>Luckily</u>: <u>Lemma</u>: These two topologies on $Hom(V,W)$ are the same!

<u>Pf</u>: We may as well assume that $V = \mathbb{R}^n$, $W = \mathbb{R}^m$.
Basic open sets in the compact-open topology have the form $\{\varphi : \mathbb{R}^n \to \mathbb{R}^m \mid \varphi(C) \subseteq U\}$, where C is compact and U is open.

We need to show that these sets are open in the vector space topology on $Hom(\mathbb{R}^n, \mathbb{R}^m)$. Note that $Hom(\mathbb{R}^n, \mathbb{R}^m) \cong M_{n\times m}(\mathbb{R})$, and basic open sets for this space come from allowing the entries in a matrix to vary over some open interval in $\mathbb{R}$.

If $f: \mathbb{R}^n \to \mathbb{R}^m$ lies in $A = \{\varphi \mid \varphi(C) \subseteq U\}$, then $f(C) \subseteq U$ and since $f(C)$ is compact, $\exists$ $\varepsilon > 0$ s.t. $B_\varepsilon(f(C)) \subseteq U$. We will show that $\exists\, \delta$ s.t. if $g: \mathbb{R}^n \to \mathbb{R}^m$ satisfies $|\langle g(e_i), e_j\rangle - \langle f(e_i), e_j\rangle| < \delta$ for all i, j (meaning that the matrix entries for f and g are all within δ of one another) then $g(C) \subseteq U$ as well, i.e. $g \in A$. This will show that A is open in the vector space topology.

Since C is compact, $C \subseteq B_r(0)$ for some r, & for any $x = \sum x_i e_i \in C$, we have $|x_i| \le r$. Set $\delta = \varepsilon / rmn$.

Then if $|\langle ge_i, e_j\rangle, \langle fe_i, e_j\rangle| < \delta$ $\forall\, i, j$, we have:

$|g(x) - f(x)| < \varepsilon$ for all $x \in C \subseteq B_r(0)$: if $x = \sum x_i e_i$,

$$|g(x) - f(x)| \le \sum_{i=1}^{n} |x_i|\, |g(e_i) - f(e_i)| \le \sum_{i=1}^{n} r \left| \sum_{j=1}^{m} \langle ge_i, e_j\rangle e_j - \sum_{j=1}^{m} \langle fe_i, e_j\rangle e_j \right|$$

$$\le \sum_{i,j} r\, |\langle ge_i, e_j\rangle - \langle fe_i, e_j\rangle| \le r \cdot n \cdot m \delta \le \varepsilon$$

So $g(C) \subseteq B_\varepsilon(f(C)) \subseteq U$.

In the other direction, we need to show that $\{\varphi:\mathbb{R}^n\to\mathbb{R}^n \mid \forall_{ij} \langle \varphi e_i, e_j\rangle \in \mathcal{U}_{ij}\}$ $\underbrace{\text{is a compact-open}}_{\text{i.e. open in the C-O topology}}$ set ($\forall\ \mathcal{U}_{ij}$ open in $\mathbb{R}$).

This set is the intersection of $\{\varphi \mid \langle \varphi e_{ij}, e_j\rangle \in \mathcal{U}_{ij}\}$ over i and j, so it suffices to show these simpler sets are compact-open.

But $\{\varphi \mid \langle \varphi e_i, e_j\rangle \in \mathcal{U}_{ij}\} = \{\varphi \mid \varphi(e_i) \subseteq \mathbb{R}\times\cdots\times \underset{j^{th}\text{ spot}}{\mathcal{U}_{ij}} \times\cdots\times\mathbb{R}\}$

and since $\{e_i\}$ is compact and $\mathbb{R}\times\cdots\times\mathcal{U}_{ij}\times\cdots\times\mathbb{R}$ is open, this finishes the proof. $\square$

<u>Corollary:</u> A function $\varphi: \mathcal{U}\times\mathbb{R}^n \to \mathcal{U}\times\mathbb{R}^n$ is continuous if and only if its "adjoint"

$$\begin{aligned} \varphi^\vee : \mathcal{U} &\longrightarrow \mathrm{Isom}(\mathbb{R}^n,\mathbb{R}^n) \\ u &\longmapsto \varphi_u : \{u\}\times\mathbb{R}^n \to \{u\}\times\mathbb{R}^n \end{aligned}$$

is continuous.

<u>Pf</u>: φ is cont. $\iff$ $\begin{aligned} \mathcal{U}\times\mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ u,\vec{v} &\longmapsto \varphi(u)(\vec{v}) \end{aligned}$ is continuous.

In general, if W is locally compact and regular (e.g. $W=\mathbb{R}^n$) then a function $\mathcal{U}\to \mathrm{Map}(W,V)$ is continuous if and only if the adjoint $\mathcal{U}\times W\to V$ is continuous. (This basic property of the compact-open topology is proven in Munkres. The "if" direction holds for <u>any</u> space W.) $\square$

Given such a cont. functor F, we want to extend F to a functor

$$\underline{Vect}(X)^k \longrightarrow \underline{Vect}(X),$$

where $\underline{Vect}(X)$ is the category of v. bdles over X.

<u>Defn</u>: Given $E_1, \dots, E_k \in \underline{Vect}(X)$ and a cont. functor $F : \underline{Vect}^k \to \underline{Vect}$, we define

$$F(E_1, \dots, E_k) = \bigcup_{x \in X} F(p_1^{-1}(x), \dots, p_k^{-1}(x)) \xrightarrow{p} X$$

$$F(\{p_i^{-1}(x)\}) \longmapsto x$$

(here $E_i \xrightarrow{p_i} X$ are the projections), with the following topology.

Say $U \subseteq X$ is an open set over which each E_i is trivial, and let $\varphi_i : U \times \mathbb{R}^{n_i} \xrightarrow{\cong} E_i|_U$ be trivializations, and let $\varphi_{ix} : \{x\} \times \mathbb{R}^{n_i} \to E_i|_{\{x\}}$ be the restrictions. Then we have $F(\varphi_{1x}, \dots, \varphi_{kx}) : F(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}, \dots, \mathbb{R}^{n_k}) \xrightarrow{\cong} F(E_1|_x, \dots, E_n|_x)$, which assemble to a fuction

$$\varphi : U \times F(\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_k}) \longrightarrow F(E_1, \dots, E_k)|_U.$$

$$(x, \alpha) \longmapsto F(\varphi_{1x}, \dots, \varphi_{kx})(\alpha)$$

We declare a set $A \subset F(E_1, \ldots, E_k)$ to be open if and only if each preimage $\varphi^{-1}(A)$ is open in $U \times F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k})$. Since $\varphi^{-1}(\bigcup A_i) = \bigcup \varphi^{-1}(A_i)$ and $\varphi^{-1}(A_1 \cap \cdots \cap A_\ell) = \bigcap_{i=1}^{\ell} \varphi^{-1}(A_i)$, this is a topology.

Lemma: $\begin{array}{c} F(E_1, \ldots, E_k) \\ \downarrow p \\ X \end{array}$ is a (locally trivial) vector bdle.

Pf: To check that p is continuous, it suffices to check that its restriction to each $F(E_1, \ldots, E_k)|_U$ (U a set over which all the E_i are trivial) is continuous. But this is immediate.

Next we will check that each map

$$\varphi : U \times F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k}) \longrightarrow F(E_1, \ldots, E_k)|_U$$

considered above is a homeomorphism. This means we must check that if $W \subset F(E_1, \ldots, E_k)|_U$ is open, then so is $\psi(\varphi^{-1}(W)) \subset U' \times F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k})$ (where ψ is any other such trivialization). In other words, we must check that $\psi^{-1}\varphi$ is continuous. This follows from continuity of F: since F is a functor, the maps

$$U \to \mathrm{Isom}(F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k}), F(\mathbb{R}^{n_1}, \ldots, \mathbb{R}^{n_k}))$$
$$u \mapsto F(\psi_{1u}^{-1}\varphi_{1u}, \ldots, \psi_{ku}^{-1}\varphi_{ku})$$

and

$$u \longmapsto F(\psi_{1u}, \ldots, \psi_{ku})^{-1} \circ F(\varphi_{1u}, \ldots, \varphi_{ku})$$

are the same, and the latter has adjoint equal to $\psi^{-1}\varphi$.

By the Lemma, to check continuity of $\psi^{-1}\varphi$ we can just check continuity of $u \mapsto F(\psi_{1u}^{-1}\varphi_{1u}, \dots, \psi_{ku}^{-1}\varphi_{ku})$. We construct this map as follows:

Start with the maps $U \times \mathbb{R}^{n_i} \xrightarrow{\psi_i^{-1}\varphi_i} U \times \mathbb{R}^{n_i}$,

and consider their adjoints $U \xrightarrow{(\psi_i^{-1}\varphi_i)^\vee} \mathrm{Isom}(\mathbb{R}^{n_i}, \mathbb{R}^{n_i})$.

These are continuous (by the Lemma) and hence so is

$$\begin{aligned} U &\longrightarrow \mathrm{Isom}(\mathbb{R}^{n_1}, \mathbb{R}^{n_1}) \times \dots \times \mathrm{Isom}(\mathbb{R}^{n_k}, \mathbb{R}^{n_k}) \\ u &\longmapsto \Big(\underbrace{(\psi_1^{-1}\varphi_1)^\vee(u)}_{= \psi_{1u}^{-1}\varphi_{1u}}, \dots, \underbrace{(\psi_k^{-1}\varphi_k)^\vee(u)}_{= \psi_{ku}^{-1}\varphi_{ku}} \Big) \end{aligned}$$

Applying the functor F gives the map

$$\begin{aligned} U &\longrightarrow \mathrm{Isom}(F(\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_k}), F(\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_k})) \\ u &\longmapsto F(\psi_{1u}^{-1}\varphi_{1u}, \dots, \psi_{ku}^{-1}\varphi_{ku}) \end{aligned}$$

which is continuous by continuity of F.

This is exactly what we wanted to show. $\square$