

Lecture 10

Cohomology of Projective Spaces

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Before addressing the Projective Bdd Thm, we consider the case of computing $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$ and $H^*(\mathbb{C}P^n; \mathbb{Z})$.

Prop'n: For each n , $\mathbb{R}P^n$ is a CW cplx with one e_k in each dimension $1 \leq k \leq n$. The $(n-1)$ -skeleton of $\mathbb{R}P^n$ is precisely $\mathbb{R}P^{n-1}$, and the attaching map $\psi_n: S^{n-1} \rightarrow \mathbb{R}P^{n-1} = (\mathbb{R}P^{n-1})^{(n-1)}$ is just the quotient map defining $\mathbb{R}P^{n-1}$.

PF: By induction on n . For $\mathbb{R}P^1 = S^1$, there is nothing to prove. Now, $\mathbb{R}P^n = S^n / x \sim -x$. If $D^n \hookrightarrow S^n$ is inclusion of the upper hemisphere, then the induced map $D^n /_{\substack{x \sim -x \\ \text{for } x \in \partial D^n}} \rightarrow S^n /_{x \sim -x} = \mathbb{R}P^n$ is a homeomorphism. But the subspace $\partial D^n /_{x \sim -x} \subseteq D^n /_{\substack{x \sim -x \\ \text{for } x \in \partial D^n}}$ is precisely $\mathbb{R}P^{n-1}$! So $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup_{\partial D^n} D^n$ and the attaching map is as described. \square

Corollary 1: $H^k(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for $0 \leq k \leq n$, and is 0 otherwise.

Proof: Since $H^k(X; \mathbb{Z}/2) \cong \text{Hom}(H_k(X; \mathbb{Z}/2), \mathbb{Z}/2)$. (Univ. Coeff. Thm.) it suffices to compute $H_k(\mathbb{R}P^n; \mathbb{Z}/2)$. We will compute

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$H_*(\mathbb{R}P^n; \mathbb{Z}/2)$ via cellular homology. The cellular chain complex has the form

$$0 \rightarrow \mathbb{Z}/2 \xrightarrow{d_n} \mathbb{Z}/2 \xrightarrow{d_{n-1}} \mathbb{Z}/2 \rightarrow \dots \xrightarrow{d_2} \mathbb{Z}/2 \xrightarrow{d_1} \mathbb{Z}/2 \rightarrow 0,$$

where d_k is the cellular boundary map. It is simply multiplication by the degree (mod 2) of the composite

$$S^{k-1} \xrightarrow{\eta_k} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1} / \mathbb{R}P^{k-2} \xrightarrow{\cong} D^{k-1} / \partial D^{k-1} = S^{k-1}$$

where $\varphi_{k-1}: D^{k-1} \rightarrow \mathbb{R}P^n$ is the characteristic map for the $(k-1)$ -cell.

This map restricts to homeomorphisms $S_+^{k-1} \xrightarrow{\cong} D^{k-1} - \partial D^{k-1}$ and $S_-^{k-1} \xrightarrow{\cong} D^{k-1} - \partial D^{k-1}$ (S_{\pm}^{k-1} the hemispheres) so the degree is the (signed) number of inverse images of any $x \in D^{k-1} - \partial D^{k-1}$, which is $(\pm 1) + (\pm 1) = 0$ or $\pm 2 = 0 \pmod{2}$. So all the d_k are 0, and we have $H_k(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2$ for $0 \leq k \leq n$, and $H^k(\mathbb{R}P^n; \mathbb{Z}/2) = \text{Hom}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$ for $0 \leq k \leq n$. □

Before computing the ring structure, we consider the complex case.

Prop'n: $\mathbb{C}P^n$ is a CW complex with one cell in each even dimension $(0, 2, 4, \dots, 2n)$. The $(2n-2)$ -skeleton of $\mathbb{C}P^n$ is $\mathbb{C}P^{n-1}$, and the $2n$ cell is attached via $\partial D^{2n} = S^{2n-1} \xrightarrow{q} \mathbb{C}P^{n-1}$, the quotient map.

Proof: $\mathbb{C}P^n = S^{2n+1} / \sim$ for $\lambda \in S^1 \subset \mathbb{C}$, and we need

to view this as D^{2n} / \sim for $v \sim \lambda v$ for $v \in \partial D^{2n} = S^{2n-1}$; the latter

is precisely a $2n$ -cell attached to $\mathbb{C}P^{n-1} = S^{2n-1} / \sim$

via the quotient map. The disk we will use is

$D_+^{2n} = \{(z_1, \dots, z_{n+1}) \in S^{2n+1} \subset \mathbb{C}^{n+1} \mid z_{n+1} \in \mathbb{R}_{\geq 0}\}$, which is

just a hemisphere. The map $D_+^{2n} \rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n$ is

onto, and it's injective on $\{(z_1, \dots, z_{n+1}) \in D_+^{2n} \mid z_{n+1} > 0\}$.

(b/c under rotation by $\lambda = e^{i\theta} \in S^1$, each $z \in \mathbb{C} \setminus \{0\}$ gets identified with exactly one pt. on $\mathbb{R}_{>0}$). So $D_+^{2n} / \sim \cong \mathbb{C}P^n$. \square

Corollary: $H_*(\mathbb{C}P^\infty; \mathbb{Z}) = H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & * \text{ even} \\ 0, & * \text{ odd} \end{cases}$.

PF: The cellular chain cplx has the form

$$0 \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \dots \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0$$

b/c all the cells are in even dim's. So the bdy maps are

automatically zero. The same goes for cohomology. \square

The cup product structure on $H^*(\mathbb{R}P^n)$, $H^*(\mathbb{C}P^n)$ can be computed via somewhat delicate geometric arguments (see Hatcher) but can also be deduced from Poincaré Duality.

Prop (Hatcher 3.38): The cup product pairing

$$(\star) \quad H^k(M; \mathbb{Z}) \times H^{n-k}(M; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$\varphi, \psi \longmapsto (\varphi \cup \psi)[M]$$

is non-singular for any orientable mfd M with $H^*(M; \mathbb{Z})$ torsion-free.

Similarly, the corresponding pairing with $\mathbb{Z}/2$ -coeffs is non-singular for any mfd.

Rmk: Such a bilinear pairing $A \times B \rightarrow R$ is called non-singular if the induced maps

$$A \rightarrow \text{Hom}(B, R), \quad B \rightarrow \text{Hom}(A, R)$$

are isomorphisms.

Proof: We can factor the map $H^k \rightarrow \text{Hom}(H^{n-k}, \mathbb{Z})$
 $\varphi \mapsto (\psi \mapsto (\varphi \cup \psi)[M])$

as follows (up to sign):

$$H^k(M; \mathbb{Z}) \xrightarrow{h} \text{Hom}(H_k(M; \mathbb{Z}), \mathbb{Z}) \xrightarrow{D^*} \text{Hom}(H^{n-k}(M; \mathbb{Z}), \mathbb{Z})$$

$$\varphi \mapsto (\sigma \mapsto \varphi(\sigma)) \quad f \mapsto (\psi \mapsto f(\psi \cap [M]))$$

b/c the composite is $\varphi \mapsto (\psi \mapsto \varphi(\psi \cap [M]) = \pm(\varphi \cup \psi)[M])$. The first map is an isom. by the UCT, the 2nd by Poincaré Duality. The $\mathbb{Z}/2$ case is the same. Switching k and $n-k$ completes the proof. \square

Corollary: $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$, with $|\alpha| = 2$

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ring isomorphism
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$H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/(\alpha^{n+1})$, with $|\alpha| = 1$.

Pf: There is a ring map $\mathbb{Z}[\alpha]/(\alpha^{n+1}) \rightarrow H^*(\mathbb{C}P^n; \mathbb{Z})$, defined by sending $\alpha \mapsto \alpha = c_1(\gamma_1) \in H^2(\mathbb{C}P^n; \mathbb{Z})$ (the canonical generator). By our additive computation of $H^*(\mathbb{C}P^n; \mathbb{Z})$, it will suffice to show that $\alpha^k \in H^{2k}(\mathbb{C}P^n; \mathbb{Z})$ is a generator (for $k = 2, \dots, n$). We prove this by induction on n . For $n = 1$, there is nothing more to prove, since we know α generates $H^2(\mathbb{C}P^1; \mathbb{Z})$. The inclusions $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ induce isomorphisms $H^k \mathbb{C}P^{n-1} \xrightarrow{\cong} H^k \mathbb{C}P^n$ for $k \leq 2n-2$, since $\mathbb{C}P^n$ is formed from $\mathbb{C}P^{n-1}$ by attaching a $2n$ -cell. (In the real case, $H^k(\mathbb{R}P^{n-1}; \mathbb{Z}/2) \xrightarrow{\cong} H^k(\mathbb{R}P^n; \mathbb{Z}/2)$ for $k \leq n-1$, because the cellular boundary maps in $C^*(\mathbb{R}P^n; \mathbb{Z}/2)$ are 0, so $C_* \cong H_*$.) So by induction we may assume that α^k generates $H^{2k}(\mathbb{C}P^n; \mathbb{Z})$ for $k < n$.

By the Prop'n, the homomorphism $H^2(\mathbb{C}P^n; \mathbb{Z}) \xrightarrow{f} \mathbb{Z}$
 $\alpha \mapsto 1$

is given by $\psi \mapsto (\psi \cup \psi [\mathbb{C}P^n])$ for some class $\psi \in H^{2n-2}(\mathbb{C}P^n; \mathbb{Z})$. So

$$1 = f(\alpha) = \psi \cup \alpha [\mathbb{C}P^n],$$

which means that neither ψ nor $\psi \cup \alpha$ can be written as $m\beta$ with $m > 1, \beta \in H^*(\mathbb{C}P^n; \mathbb{Z})$. Then ψ generates $H^{2n-2}(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$, and we have $\psi = \pm \alpha^{n-1}$ by induction, and similarly $\psi \cup \alpha$ generates $H^{2n}(\mathbb{C}P^n; \mathbb{Z})$.

Now $\psi \cup \alpha = \pm \alpha^{n-1} \cup \alpha = \pm \alpha^n$, so α^n generates $H^{2n}(\mathbb{C}P^n; \mathbb{Z})$. \square