Conclusion to Proof of Bott Periodicity

<u>Lemma 2.8</u>: Let $b: E \to E$ be an endomorphism s.t. $\forall x$, $b_x : E_x \to E_x$ has no e. value in S^1.

Let $(E_x)_+$ = direct sum of generalized eigenspaces for b_x w/ e. value of length <1, $= \bigoplus_{|\lambda|<1} \text{g.e.s.}(b_x, \lambda)$

and let $(E_x)_- = \bigoplus_{|\lambda|>1} \text{g.e.s.}(b_x, \lambda)$.

Then
$$E_x = (E_x)_+ \oplus (E_x)_-,$$
and in fact, we have a Whitney sum decomp.
$$E = E_+ \oplus E_-,$$
where $E_+ = \bigcup_{x \in X} (E_x)_+$ and $E_- = \bigcup_{x \in X} (E_x)_-$

<u>Pf</u>: We just need to show that E_+ and E_- are locally trivial. We may work locally, and hence we can assume E is trivial and b is represented by a map $b: X \to M_n\mathbb{C}$.

Let $q_x(t)$ be the characteristic polynomial of $b_x = b(x) \in M_n\mathbb{C}$. Then
$$q_x(t) = q_x^-(t)\, q_x^+(t),$$
where all roots of q_x^- lie inside S^1, and all roots of q_x^+ lie outside S^1.

Since we have $E_x = (E_x)_+ \oplus (E_x)_-$,

we can write $b_x = b_x^+ \oplus b_x^-$, where $b_x^+ : (E_x)_+ ⟲$
$b_x^- : (E_x)_- ⟲$.

<u>Notice</u>: the characteristic polynomial of

b_x^- is precisely q_x^-, and the char. poly. of

b_x^+ is q_x^+. By the Cayley-Hamilton Theorem,

$q_x^+(b_x^+) : (E_x)_+ \to (E_x)_+$ is zero, and hence

$q_x^+(b_x) : E_x \to E_x$ satisfies

(1) $$(E_x)_+ \subseteq \ker(q_x^+(b_x)).$$

Similarly,

(2) $$(E_x)_- \subseteq \ker(q_x^-(b_x)).$$

We claim that there are in fact equalities:

indeed, q_x^+ and q_x^- have no common roots, so they're

relatively prime in $\mathbb{C}[t]$. Hence $\exists\ r_x(t), s_x(t) \in \mathbb{C}[t]$

s.t. $s_x q_x^+ + r_x q_x^- = 1$, and now

$$s_x(b_x)\, q_x^+(b_x) + r_x(b_x)\, q_x^-(b_x) = Id_{E_x}.$$

Hence $\ker(q_x^+(b_x)) \cap \ker(q_x^-(b_x)) = \{0\}$.

Since $(E_x)_+ \oplus (E_x)_- = E_x$, this shows that (1), (2) are equalities.

<u>Claim:</u> $(E_x)_+ = \mathrm{Im}\,(q_x^-(b_x))$

$(E_x)_- = \mathrm{Im}\,(q_x^+(b_x))$.

<u>Pf:</u> $q_x^+(b_x)\,q_x^-(b_x) = q_x(b_x) = 0$ by Cayley-Hamilton,

so $\mathrm{Im}(q_x^- b_x) \subseteq \mathrm{Ker}(q_x^+(b_x)) = (E_x)_+$

and by commutativity, $\mathrm{Im}(q_x^-(b_x)) \subseteq \mathrm{Ker}(q_x^+(b_x)) = (E_x)_-$.

But $rq_+ + q_-s = 1$, so if $v \in \ker(q_+^+ b_x)$ then

$$\underbrace{\cancel{r(b_x)\,q_+(b_x)\,v}}_{0} + q_-(b_x)\,s(b_x)\,v = v$$

$$\Rightarrow \quad v \in \mathrm{Im}(q(b_x)). \qquad \square$$

Now, if $\{v_1, \dots, v_k\}$ is a basis for $(E_x)_+$, then $\exists\, w_1, \dots, w_k$ s.t. $v_i = q_x^-(b_x)\,w_i$, and if $\{v_{k+1}, \dots, v_n\}$ is a basis for $(E_x)_-$, $\exists\, w_{k+1}, \dots, w_n$ s.t. $v_i = q_x^+(b_x)\,w_i$.

We claim that the coeff's of the poly's q_x^+, q_x^- vary continuously w/ x. This will imply that the matrices $q_x^+(b_x)$, $q_x^-(b_x)$ vary continuously as well. Then for y close enough to x, $\{q_y^+(b_y)\,w_i\}_{i=1}^n$ is still linearly independent, and hence give bases for $(E_y)_+$ and $(E_y)_-$. This will complete the proof.

Continuity of $q_x^+ b_x$, $q_x^- b_x$ comes from the following picture:

$$\begin{tikzcd}
M_n\mathbb{C} \arrow[r, "\text{coeff's}", "\text{of } \det(b_x-\lambda I)"'] & \mathbb{C}^n \arrow[r, "\text{roots}"] \arrow[d, phantom, "\boxed{\text{monic poly's of degree} \le n}"] & \mathrm{Sym}^n\mathbb{C} = \overbrace{\mathbb{C}\times\cdots\times\mathbb{C}}^{n}/\underset{\uparrow\ \text{Symmetric group}}{\Sigma_n} \\
\cup & & \\
\left\{\begin{array}{c} B\in M_n\mathbb{C} \\ \text{s.t. } B \text{ has} \\ \text{no e.val's} \\ \text{in } S^1 \end{array}\right\} \arrow[r] \arrow[u, phantom] & W \arrow[uu, hook] \arrow[r, "\text{roots}"] & \mathrm{Sym}^n(\mathbb{C}\setminus S^1) \arrow[uu, hook]
\end{tikzcd}$$

$$W \leftrightarrow \boxed{\text{monic poly's w/o roots in } S^1}$$

$$q \colon \left\{\begin{array}{c} B\in M_n\mathbb{C} \\ \text{s.t. } B \text{ has no e.val's in } S^1 \end{array}\right\} \longrightarrow (q^+, q^-)$$

$$\mathrm{Sym}^n(\mathbb{C}\setminus S^1) = \coprod_{k=0}^{n} \mathrm{Sym}^k(D_0)\times \mathrm{Sym}^{n-k}(D_\infty)$$

$$[\lambda_1,\dots,\lambda_k],\ [\gamma_1,\dots,\gamma_n] \mapsto \prod_{i=1}^{k}(t-\lambda_i),\ \prod_{i=1}^{n}(\gamma-\lambda_i)$$

$$(q^+, q^-) \in \coprod_{k=0}^{n} \left\{\begin{array}{c}\text{monic poly's} \\ \text{of degree } k \text{ w/} \\ \text{all roots inside } S^1\end{array}\right\} \times \left\{\begin{array}{c}\text{monic poly's} \\ \text{of deg } n-k \\ \text{w/ all roots outside } S^1\end{array}\right\}$$

$$\cap$$

$$\coprod_{k=0}^{n} \mathbb{C}^k \times \mathbb{C}^{n-k}$$

again, we think of monic poly's as elt's of $\mathbb{C}^k$ via their coeff's.

Continuity of the map "roots" follows from Rouché's Theorem.

Conclusion of Proof of Bott Periodicity:

We've now shown that for any finite CW cplx X,

$$\begin{aligned} K^0(X)\times K^0(S^2) &\xrightarrow{\mu} K^0(X\times S^2) \\ \alpha\ ,\ \xi &\longmapsto \pi_1^*\alpha\otimes\pi_2^*\xi \end{aligned}$$

is surjective. We want to show that μ is injective, as well as that the Bott map

$$\begin{aligned} \beta: \tilde{K}^0(X) &\longrightarrow \tilde{K}^0(S^2X) \\ \alpha &\longmapsto \alpha\otimes(\pi_2^*H-1) \end{aligned}$$

is an isomorphism. (The latter is Bott Periodicity.)

Let's recall how β and μ are related:

$$\begin{tikzcd}[column sep=small]
K^0(X)\otimes K^0(S^2) \arrow[d, "\mu"] & =(\tilde{K}^0(X)\oplus\mathbb{Z})\otimes(\tilde{K}^0(S^2)\oplus\mathbb{Z})\cong \tilde{K}^0(X)\otimes\overbrace{\tilde{K}^0(S^2)}^{\text{gen'd by } H-1} \arrow[d, "\beta"] & \oplus\tilde{K}^0(X)\oplus\tilde{K}^0(S^2) \arrow[d, "="] & \oplus\mathbb{Z} \arrow[d, "="] \\
K^0(X\times S^2) & \cong\tilde{K}^0(X\times S^2)\oplus\mathbb{Z}\cong\underbrace{\tilde{K}^0(X\wedge S^2)}_{\cong\tilde{K}^0(S^2X)} & \oplus\underbrace{\tilde{K}^0(X)\oplus\tilde{K}^0(S^2)}_{\tilde{K}^0(X\vee S^2)} & \oplus\mathbb{Z}
\end{tikzcd}$$

So, μ surj $\Longrightarrow$ β surj. Using this, we'll show β is an isomorphism, and this will show μ is an isom. as well.

Theorem: β is an isom for all CW cplx's.

pf: First, consider spheres. Since β is surjective,

$$0=\tilde{K}^0(S^1)\twoheadrightarrow\tilde{K}^0(S^3)\twoheadrightarrow\cdots\twoheadrightarrow\tilde{K}^0(S^{2n+1})$$

So β is an isom. for odd spheres, where $\tilde{K}^0$ is always 0.

For even spheres, we use the Chern Character:

$$\begin{tikzcd}
\mathbb{Z} = \tilde{K}^0(S^0) \arrow[r, "\beta"] \arrow[d, "\cong"', "ch_0"] & \tilde{K}^0(S^2) \arrow[r, "\beta"] \arrow[d, "ch_2"] & \tilde{K}^0(S^4) \arrow[r] \arrow[d, "ch_4"] & \cdots \\
\mathbb{Z} = \tilde{H}^{even}(S^0;\mathbb{Z}) \arrow[r, "\sigma", "\cong"'] & \tilde{H}^{even}(S^2;\mathbb{Z}) \arrow[r, "\sigma", "\cong"'] & \tilde{H}^{even}(S^4;\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

suspension isom

We saw earlier that the diagram commutes, which implies that Ch has image inside $\tilde{H}^{even}(S^{2k};\mathbb{Z})$ (b/c this is clearly true when $k=0$).

Commutativity implies, inductively, that β is injective (hence an isom) and then that Ch is an isom, which allows the induction to continue. This shows β is an isom for all spheres.

Say X is any finite CW cplx. We'll show by induction on # cells in X that $\beta : \tilde{K}^0(X) \xrightarrow{\cong} \tilde{K}^0(S^2X)$. Write $X = A \cup e^k$ w/ e^k a top dim'l cell. The Puppe Seq. for $A \subset X \to X/A = S^k$ gives a diagram

$$\begin{tikzcd}
\tilde{K}^0(\Sigma A) \arrow[r] \arrow[d, "\cong"', "\beta"] & \tilde{K}^0(X/A) \arrow[r] \arrow[d, "\cong"', "\beta_{X/A}"] & \tilde{K}^0(X) \arrow[r] \arrow[d, "\beta_X"] & \tilde{K}^0(A) \arrow[d, "\cong"', "\beta_A"] \\
\tilde{K}^0(S^3A) \arrow[r] & \tilde{K}^0(S^2(X/A)) \arrow[r] & \tilde{K}^0(S^2X) \arrow[r] & \tilde{K}^0(S^2A)
\end{tikzcd}$$

isom by induction, b/c up to htpy SA has same # of cells as A.

isom b/c $X/A = S^k$

The 4½ lemma implies that β_X is an isomorphism. □