Math 191, Fall 2009
Methods for Computing Limits

This sheet lists some of the methods we have learned for computing limits, and describes when they apply. If you don't remember how to use one of these methods, look it up in the textbook.

If you're lucky, every limit you'll see will immediately suggest to you one of these methods, and you'll go straight to the solution. But you're quite likely to find that sometimes the first method you try doesn't work. Don't give up! If one method fails, try another.

We'll discuss a hypothetical limit $\lim_{x\to a} f(x)$.

1) **Continuity.** If $f(x)$ is continuous at $x = a$, then $\lim_{x\to a} f(x) = f(a)$. Begin by checking wether this method applies. You'll often complete a problem by using this method, after using other methods.

This method fails for many piecewise functions, i.e. when $f(x)$ is given by different formulas depending on the value of x.

If your function has an infinite discontinuity (e.g. $f(x) = 1/x$ with $a = 0$, or $f(x) = 1/|x|$ with $a = 0$), the limit will either not exist, or be equal to ∞, or be equal to $-\infty$ (think about this!).

This method also fails when computing $f(a)$ leads to an indeterminate form:

Examples:

- $f(x) = \frac{\sin(x)}{1-\cos(x)}$: at $x = 0$, $f(x)$ is indeterminate of type $\frac{0}{0}$.
- $f(x) = \frac{1}{x-1} - \frac{2}{x^2-1}$: at $x = 1$, $f(x)$ is indeterminate of type $\infty - \infty$.
- $f(x) = \frac{x^2-6x+9}{x-3}$: at $x = 3$, $f(x)$ is indeterminate of type $\frac{0}{0}$.

If your limit cannot be evaluated by continuity, try another method. The following methods are not in any particular order.

2) **Cancel common factors in fractions:** This method is often used after beginning with another method. But sometimes it works right away, especially if $f(x)$ is indeterminate of type $\frac{0}{0}$ at $x = a$. Try this with the 3rd example above. Notice that in the end, you'll use Method 1) to complete the problem.

In many of the examples you'll encounter, the key to solving the problem will be to apply this method at some point, in order to cancel an expression involving x from the top and bottom of a fraction. You should always keep this method in the back of your mind as you work on a problem.

3) **Combine fractions:** If you see a limit involving a difference of fractions, it can sometimes be helpful to combine them over a common denominator. Try this with the second example above (it's quickest if you're careful: what's the least common multiple of $x - 1$ and $x^2 - 1$?). Notice that, in working through this example, you eventually use Method 2), and you finish with Method 1).

4) **Multiply by the conjugate:** This mainly applies to limits involving square roots, although we used an analogous idea to compute $\lim_{x\to 0} \frac{1-\cos(x)}{x}$: we mul-

tiplied top and bottom by $1+\cos(x)$, which turned the function into something related to $\frac{\sin(x)}{x}$ (so we were using Method 6) below).

5) **The Squeeze Theorem:** This is in some sense one of the harder methods to apply, but once you get the hang of it, all the examples start to look pretty similar! The Squeeze Theorem should only be used in situations where the previous methods don't apply. We tend to use it in cases involving sin and cos, where the argument (what you're plugging in to sin or cos) is tending to infinity (causing the function to oscillate wildly).

To use this method, you need bounds on your function, and the starting point is usually the inequalities

$$-1 \leqslant \sin(\theta) \leqslant 1, \qquad -1 \leqslant \cos(\theta) \leqslant 1.$$

Note that even if θ is a complicated function of x, these bounds still apply! For example,

$$-1 \leqslant \sin\left(\frac{e^x}{\ln(|x|+3)}\right) \leqslant 1.$$

You can build up one step at a time to find bounds for more complex functions.

Sometimes it's easiest to work with the left- and right-handed limits separately, since different bounds may apply (this often gets around the confusing business of putting in absolute value signs; see the solutions to Quiz 5). Try these examples (if you do them in order, you should start to see that the later ones are not really much harder than the first!). One of them is a trick example: Method 1) actually applies.

- $\lim_{x\to 1} |x-1| \cos\left(\frac{1}{x-1}\right)$
- $\lim_{x\to 1} (x-1) \cos\left(\frac{1}{(x-1)^3}\right)$ [Try the left and right limits separately.]
- $\lim_{x\to 1} (x-1)^5 \sin\left(\cos\left(\frac{1}{(x-1)^3}\right)\right)$
- $\lim_{x\to 1} (x-1)^5\, 2\sin\left(\cos\left(\frac{1}{(x-1)^3}\right)\right)$
- $\lim_{x\to 1} (x-1)^5\, 2\sin\left(\cos\left(\frac{1}{(x-2)^3}\right)\right)$
- $\lim_{x\to 1} (x-1)^5\, e^{2\sin\left(\cos\left(\frac{1}{(x-1)^3}\right)\right)}$

6) **Compare with a limit you know:** If the previous methods fail, you may be able to compare your limit with a known limit, such as

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \quad \text{or} \quad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0.$$

Examples 9-40 in Section 2.6 are of this type.

7) **Combine methods:** Some problems require multiple methods. For example, in a Squeeze Theorem problem like those above, you might have more complicated bounds, and another method might have to be used to calculate the limits of the bounding functions. For example, replace $(x-1)^5$ by $\left|\frac{1-\cos(x)}{x}\right|$ in the Squeeze Theorem examples above, and then you'll need Method 6).