Solutions to Exam 2

[15 points] 1. Calculate the derivative of each of the following functions. It is not necessary to simplify your answers.

a) $g(x) = (x^4 - 3x^2 + 1)(x - \sqrt{x} + 7x^{-2})$

$$g'(x) = (4x^3 - 6x + 1)(x - \sqrt{x} + 7x^{-2}) + (x^4 - 3x^2 + 1)(1 - \tfrac{1}{2}x^{-1/2} - 14x^{-3})$$

b) $f(t) = \dfrac{e^t + t^2}{t^4 - 5t + 8}$ $\quad f'(t) = \dfrac{(e^t + 2t)(t^4 - 5t + 8) - (e^t + t^2)(4t^3 - 5)}{(e^t + t^2)^2}$

c) $y = \ln(\sin^{-1} x^2)$ $\quad y' = \dfrac{2x}{\sin^{-1}(x^2)\sqrt{1 - x^4}}$

[10 points] 2. Calculate the derivative of each of the following functions. It is not necessary to simplify your answers.

a) $f(x) = \sin(e^x)$ $\quad f'(x) = \cos(e^x) \cdot e^x$

b) $y = \tan^{-1}(x^3 + 1)$ $\quad y' = \dfrac{3x^2}{(x^3+1)^2 + 1}$

[10 points] 3. Calculate $f''(4)$, where $f(x) = 2x^3 - 3x + 32x^{1/2}$. **SIMPLIFY YOUR ANSWER.**

$$f''(x) = 12x - 8x^{-3/2}, \text{ so } f''(4) = 48 - \frac{8}{4^{3/2}} = 48 - \frac{8}{(\sqrt{4})^3} = 47.$$

[10 points] 4. Calculate $\frac{dy}{dx}$ at $x = 1$ if $y = \sin\left(\dfrac{\pi(x+1)}{x+5}\right)$. **SIMPLIFY YOUR ANSWER.**

$$\frac{dy}{dx} = \cos\left(\frac{\pi(x+1)}{x+5}\right) \cdot \frac{\pi(x+5) - \pi(x+1)}{(x+5)^2}, \text{ so } \left.\frac{dy}{dx}\right|_{x=1} = \cos\left(\frac{2\pi}{6}\right) \cdot \frac{4\pi}{36}$$

$$= \frac{1}{2} \cdot \frac{\pi}{9} = \frac{\pi}{18}.$$

[10 points] 5. Find an equation for the tangent line to $y = x^{\sin x}$ at the point $x = \frac{\pi}{2}$.

$\ln y = \sin x \ln x$, so $\dfrac{y'}{y} = \cos x \ln x + \dfrac{\sin x}{x}$ and $y'\left(\frac{\pi}{2}\right) = (\pi/2)^1 \cdot \left(0 \cdot \ln\frac{\pi}{2} + \dfrac{1}{\pi/2}\right)$

$= 1.$

Hence the eq'n of the tangent line is $y - (\pi/2)^1 = 1(x - \pi/2)$, i.e. $\boxed{y = x}$.

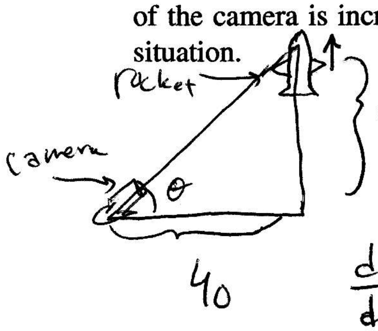
[10 points] 6. Use implicit differentiation to calculate dy/dx at (2,1) if

$$x^2y + xy^2 = x^3 - 2.$$

$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 3x^2$, so $\frac{dy}{dx} = \frac{3x^2 - y^2 - 2xy}{x^2 + 2xy}$.

Hence when $(x,y) = (2,1)$, $\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=1}} = \frac{3\cdot4 - 1 - 2\cdot2}{4 + 2\cdot2} = 7/8$.

[15 points] 7. A television camera is positioned 40 miles from the base of a rocket launching pad. The angle of elevation of the camera has to change in order to to keep the rocket in sight. If the rocket rises vertically and its speed is 6 miles/minute, find the rate at which the angle of elevation of the camera is increasing 5 seconds after lift-off. Make sure to draw a picture describing this situation.



We know that $\frac{dh}{dt} = 6\,mi/min = \frac{1}{10}\,mi/sec$, and we need to find $\frac{d\theta}{dt}$ when $t = 5$ sec. From the picture, we have $\tan\theta = h/40$, so $\theta = \tan^{-1}(h/40)$ and

$\frac{d\theta}{dt} = \frac{1}{(h/40)^2 + 1} \cdot \frac{dh/dt}{40}$. When $t = 5$ sec, $h = 5\cdot\frac{1}{10} = \frac{1}{2}$ mi,

so $\left.\frac{d\theta}{dt}\right|_{t=5\,sec} = \frac{1}{\frac{1/4}{(40)^2} + 1} \cdot \frac{1/10}{40} = \frac{1}{10\left(\frac{1/4}{40} + 40\right)} = \frac{1}{\frac{1}{16} + 400} = \frac{16}{6401}\ \frac{rad}{sec}$.

[10 points] 8. Let $f(x) = \ln x$. Estimate Δf at $a = 1$ using Linear Approximation, where $\Delta x = 0.02$.

$\Delta f \approx f'(1)\Delta x = \frac{1}{1}\cdot 0.02 = 0.02$.

[10 points] 9. Find the absolute maximum and the absolute minimum values of $f(x) = x^3 - \frac{3}{2}x^2 + 1$ on the interval $[0, 2]$.

$f'(x) = 3x^2 - 3x = 3x(x-1)$, so the critical points of f are $x = 0$ and $x = 1$. Thus the absolute max and min must occur at $x = 0$, $x = 1$, or $x = 2$.

$f(0) = 1$, $f(1) = 1 - 3/2 + 1 = 1/2$, $f(2) = 8 - \frac{3}{2}\cdot4 + 1 = 3$,

So the absolute minimum value is $1/2$ and the absolute max. is 3.