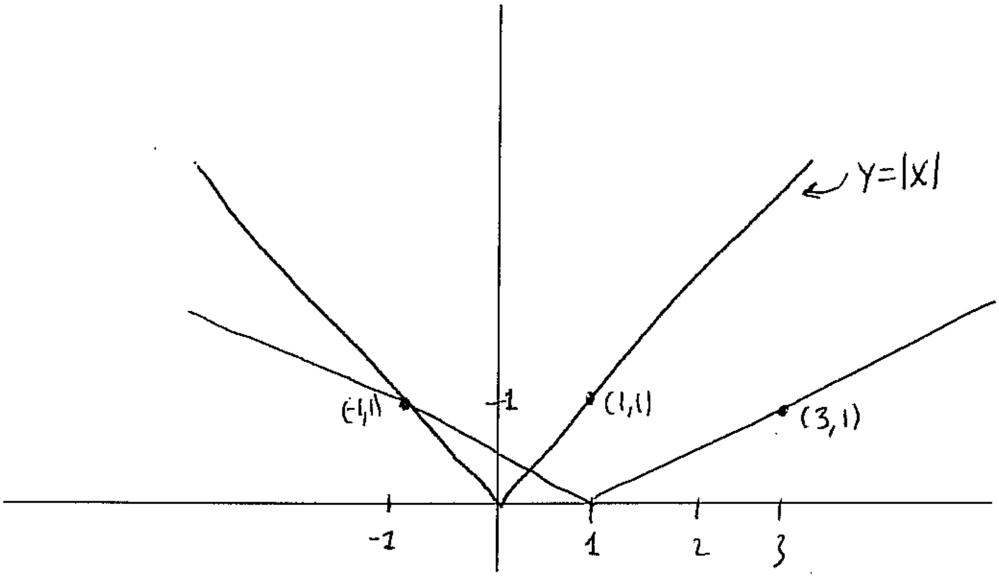
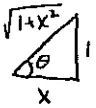
Solutions to Exam 1

[5] 1. Given that $f(x) = |x|$, sketch the graph of $y = f(x)$ and $y = \frac{1}{2}f(x-1)$ on the axis provided below. Make sure you label at least three points on the graph with their (x, y)–coordinates.



[5] 2. Simplify by referring to the appropriate triangle or trigonometric identity: $\sin(\cot^{-1} x)$.

Consider the triangle . Then $\cot(\theta) = X$, so $\theta = \cot^{-1}(x)$, and $\sin(\theta) = \frac{1}{\sqrt{1+x^2}}$. Note that since $\cot^{-1}(x) \in (0, \pi)$, $\sin(\cot^{-1}(x))$ is always positive, which agrees with our formula.

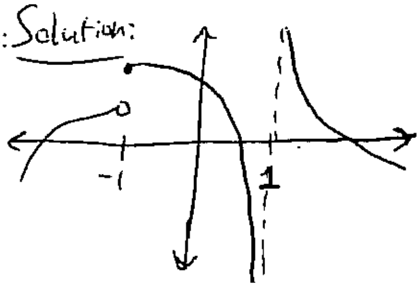
[5] 3. Find the domain of $f(x) = \frac{\sqrt{x+4}}{x^2-25}$ Since $x+4$ must be ≥ 0 and $X^2 - 25$ must not be 0, the domain is $[-4,5) \cup (5,\infty)$.

[10] 4. Draw the graph of a function $f(x)$ with all of the following properties: Solution:

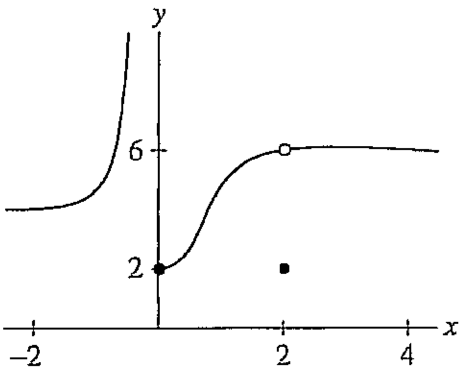
(a) $f(x)$ has a jump discontinuity at $x = -1$,

(b) $f(x)$ has an infinite discontinuity at $x = 1$,

(c) $f(x)$ is defined and continuous except at $x = -1$ and $x = 1$.



[15] 5. The graph of the $f(x)$ is shown below.



(1) Evaluate the following limits, or explain why they do not exist:

(a) $\lim_{x\to 0^-} f(x) = +\infty$

(b) $\lim_{x\to 0^+} f(x) = 2$

(c) $\lim_{x\to 2} f(x) = 6$

(2) Is $f(2)$ defined? Yes If yes, compute $f(2) = 2$

(3) State the numbers c at which f is not continuous.

$C = 0, 2$

[10] 6. Show that $x^3 + x = 3$ has a solution on the interval $(0, 2)$. **Justify your answer.**

Since $0^3+0=0<3$ and $2^3+2=10>3$, the Intermediate Value Theorem states that the continuous function $y=x^3+x$ must take on the value $y=3$ for some $x\in(0,2)$.

[25] 7. Evaluate the following limits, or explain why they do not exist. **Simplify your answers.**

(a) $\lim_{x\to 0} |x| \sin\frac{1}{x} = 0.$

Since $-1 \le \sin(\frac{1}{x}) \le 1$, we have $-|x| \le |x|\sin(1/x) \le |x|$. Since $\lim_{x\to 0}|x| = \lim_{x\to 0} -|x| = 0$, the Squeeze Theorem guarantees that $\lim_{x\to 0}|x|\sin(1/x) = 0$ also.

(b) $\lim_{h\to 0} \frac{(3+h)^{-2} - \frac{1}{9}}{h} = \lim_{h\to 0} \frac{\frac{1}{(3+h)^2} - \frac{1}{9}}{h} = \lim_{h\to 0} \frac{9-(3+h)^2}{(3+h)^2\cdot 9\cdot h} = \lim_{h\to 0} \frac{-6h-h^2}{(3+h)^2\cdot 9\cdot h}$

$= \lim_{h\to 0} \frac{-6-h}{(3+h)^2\cdot 9} = \frac{-6-0}{(3+0)^2\cdot 9} = -6/81 = -2/27.$

(c) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{3+3}$

$= 1/6$

(d) $\lim_{x\to 4} \left[\frac{1}{x-4} - \frac{8}{x^2-16}\right] = \lim_{x\to 4} \frac{(x+4)-8}{(x-4)(x+4)} = \lim_{x\to 4} \frac{x-4}{(x-4)(x+4)} = \lim_{x\to 4} \frac{1}{x+4}$

$= 1/8$

(e) $\lim_{x\to \frac{\pi}{4}} \log_2 \cos x = \log_2(\cos(\pi/4)) = \log_2(\sqrt{2}/2) = \log_2(2^{-1/2}) = -1/2$

continued ...

[10] 8. Assume that $\lim_{x\to-3} f(x) = 4$ and $\lim_{x\to-3} g(x) = -2$, compute

$$\lim_{x\to-3}\frac{x^2 f(x)}{[g(x)]^2+1} = \frac{(-3)^2\cdot 4}{(-2^2)+1} = \frac{36}{5}$$

[15] 9. Let $f(x) = x^2 + 2$.

(a) Use the definition of derivative to find $f'(3)$.

$$f'(3) = \lim_{h\to0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to0}\frac{(3+h)^2+2-(3^2+2)}{h} = \lim_{h\to0}\frac{9+6h+h^2-9}{h}$$

$$= \lim_{h\to0}\frac{6h+h^2}{h} = \lim_{h\to0} 6+h = 6$$

(b) Find the equation of the tangent line to $y = f(x)$ at $x = 3$. In point-slope form, the equation of the tangent line is $y-f(3) = f'(3)(x-3)$, i.e. $y-11 = 6(x-3)$

or $y = 6x-7$